Quantum Hall effect in higher dimensions, matrix models and fuzzy geometry

2006 J. Phys. A: Math. Gen. 39 12735

(http://iopscience.iop.org/0305-4470/39/41/S05)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 12735–12763 doi:10.1088/0305-4470/39/41/S05

Quantum Hall effect in higher dimensions, matrix models and fuzzy geometry

D Karabali[1] **and V P Nair**[2]

[1] Department of Physics and Astronomy, Lehman College of the CUNY, Bronx, NY 10468, USA
[2] Physics Department, City College of the CUNY, New York, NY 10031, USA

E-mail: dimitra.karabali@lehman.cuny.edu and vpn@sci.ccny.cuny.edu

Received 16 June 2006
Published 27 September 2006
Online at stacks.iop.org/JPhysA/39/12735

Abstract

We give a brief review of the quantum Hall effect in higher dimensions and its relation to fuzzy spaces. For a quantum Hall system, the lowest Landau level dynamics is given by a one-dimensional matrix action whose large N limit produces an effective action describing the gauge interactions of a higher dimensional quantum Hall droplet. The bulk action is a Chern–Simons type term whose anomaly is exactly cancelled by the boundary action given in terms of a chiral, gauged Wess–Zumino–Witten theory suitably generalized to higher dimensions. We argue that the gauge fields in the Chern–Simons action can be understood as parametrizing the different ways in which the large N limit of the matrix theory is taken. The possible relevance of these ideas to fuzzy gravity is explained. Other applications are also briefly discussed.

PACS numbers: 02.40.Gh, 11.15.−q, 73.43.Cd

1. Introduction

It is well known that when the number of elementary quanta involved in any process is very large, quantum dynamics can be approximated by classical dynamics; this is the celebrated correspondence principle. The classical phase space takes over the role of the quantum Hilbert space. Quantum observables, which are linear Hermitian operators on the Hilbert space, can be approximated by functions on the classical phase space. Properties of functions on the phase space can be obtained as limits of properties of operators on the Hilbert space. Keeping this idea of correspondence in mind, the general structure of a quantum field theory, describing gauge and matter fields, may be formulated as follows. We have an ambient spacetime differential manifold $\mathcal{M}$. Fields are functions (or sections of an appropriate bundle) on $\mathcal{M}$. They are also operators on the quantum Hilbert space of matter $\mathcal{H}_m$, and obey quantum conditions such as commutation rules, characterized by the deformation parameter $\hbar$. At finite $\hbar$, we have the quantum field theory; as $\hbar \to 0$, we can approximate the physics by a classical field theory.

The general correspondence principle, however, suggests a further extension of this idea and a new paradigm for physical theories. The spacetime manifold $\mathcal{M}$ itself may be viewed as an approximate method of description, obtained as the limit of some discrete Hilbert space $\mathcal{H}_s$. Thus, instead of functions on $\mathcal{M}$, physical fields are operators on $\mathcal{H}_s$. They are also operators on the Hilbert space $\mathcal{H}_m$ of the theory. A new deformation parameter θ, relevant to $\mathcal{H}_s$, may be introduced, so that, as $\theta \to 0$, we can approximate the theory in terms of functions on a smooth manifold $\mathcal{M}$. Thus, the usual quantum field theories are recovered in this limit. (A further limit, $\hbar \to 0$, would take us to the classical field theory.) In this formulation, fields are operators on $\mathcal{H}_s \otimes \mathcal{H}_m$, or we may view them as matrix-valued quantum operators, the matrices being of dimension $\dim(\mathcal{H}_s)$. Field theories can thus be regarded as limits of matrix models.

The mathematical structure that is relevant here is that of fuzzy geometry or, more generally, noncommutative geometry [1]. A fuzzy space is defined by a sequence of triples $(\mathcal{H}_N, \mathrm{Mat}_N, \Delta_N)$, where Mat_N is the matrix algebra of $(N \times N)$-matrices which act on the N-dimensional Hilbert space $\mathcal{H}_N$ and Δ_N is a matrix version of the Laplace operator. The matrices are taken to have an inner product given by, say, $\langle A, B\rangle = \frac{1}{N}\,\mathrm{Tr}(A^\dagger B)$, for arbitrary matrices A, B. In the large N limit, a matrix may be approximated by a function on some smooth manifold $\mathcal{M}$, the latter being a phase space corresponding to the Hilbert space $\mathcal{H}_N$. In this case, the deformation parameter θ is a function of N, with $\theta \to 0$ as $N \to \infty$. At finite N, we have the noncommutative algebra Mat_N, but this tends to the commutative algebra of functions on the smooth manifold $\mathcal{M}$ as $N \to \infty$. The Laplacian Δ_N is used to define the metric and related geometrical properties of the manifold $\mathcal{M}$. For example, information about the dimension of $\mathcal{M}$ is contained in the rate of growth of the degeneracy of eigenvalues of Δ_N.

Clearly, the idea of formulating field theories as matrix models on a fuzzy space is very appealing for a number of reasons. The matrix formulation gives a discretization of the field theory and therefore, at the very least, we get a regularization of the theory with a finite number of modes. This is analogous to the lattice regularization, but, in general, it is possible to preserve more symmetries in a fuzzification than in latticization [2]. Secondly, and perhaps most importantly, space, or spacetime, is being viewed as an approximation to a Hilbert space $\mathcal{H}_s$. Thus, the dynamics of spacetime geometry, in other words, gravity, can be naturally described as dynamics on the Hilbert space $\mathcal{H}_s$. The fact that the number of modes would be finite in a fuzzy formulation will ensure that we have a mathematically well-defined formulation of gravity.

It is worth recalling at this stage that fuzzy geometry is part of the more general framework of noncommutative geometry. Noncommutative geometry is a generalization of ordinary geometry, based on the following result. The algebra of complex-valued square-integrable functions on a manifold $\mathcal{M}$, with pointwise multiplication as the algebraic operation, is a commutative C^*-algebra. This C^*-algebra incorporates many of the geometrical properties of the manifold $\mathcal{M}$. Conversely, any commutative C^*-algebra can be represented by the algebra of functions on an appropriate space $\mathcal{M}$, with the geometrical properties of $\mathcal{M}$ being images of corresponding algebraic properties of the C^*-algebra. This result allows a change of point of view: we may take the algebra as the fundamental concept, the geometry being derived from it. The generalization is then to consider a noncommutative C^*-algebra; it may be taken as the analogue of an 'algebra of functions' on some noncommutative space. The mathematical properties of this noncommutative space are then implicitly defined by the properties of the algebra. This is the basic idea of noncommutative geometry [3–5].

More specifically, noncommutative geometry is given as a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is a noncommutative algebra with an involution, $\mathcal{H}$ is a Hilbert space on which we can realize the algebra $\mathcal{A}$ as bounded operators and $\mathcal{D}$ is a special operator which will characterize the geometry. In terms of such a spectral triple, the analogue of differential calculus on a manifold can be constructed. For the special case when $\mathcal{H}$ is the space of square-integrable spinor functions on a manifold $\mathcal{M}$ (technically, sections of the irreducible spinor bundle), $\mathcal{A}$ is the algebra of complex-valued smooth functions on $\mathcal{M}$ and $\mathcal{D}$ is the Dirac operator on $\mathcal{M}$ (for a particular metric and the Levi-Civita spin connection), the differential calculus constructed from the algebra is the standard differential calculus on $\mathcal{M}$. Going back to matrices, it is clear that the algebra of finite-dimensional matrices Mat_N can play the role of $\mathcal{A}$ and, hence, fuzzy geometry is a special case of noncommutative geometry. (The idea of using noncommutative geometry for gravity was suggested many years ago by Connes and others [3–6].)

While fuzzy spaces can be viewed as a regulator with real physics being eventually recovered when $N \to \infty$, the idea of fuzzy geometry goes further. One may regard the true physics as given by the theory at finite, but large N, the smooth manifold limit being a convenient simplification for calculations. After all, it is an elementary truism that, while we formulate physical theories on continuous spaces, infinite-dimensional Hilbert spaces, etc, we always deal with a finite set of measurements or even a finite number of possibilities for measurements. Therefore, it is almost tautological that physical theories, at least for the case of space being even dimensional, can be described by finite-dimensional matrix models.

Indeed, matrix models have recently appeared in a number of different contexts in physics. It was observed many years ago that one could use matrix models as a regularization of membrane theories [7]. By now this is well understood and matrices have become a standard technique for analysis of branes of different dimensions. Matrix models' descriptions of M-theory (in a certain kinematic limit) have been proposed [4, 8]. Fuzzy spaces emerge naturally as classical solutions of such models. Matrix models also appear, because they contain brane-like configurations, in elaborations of the gauge–gravity duality [9]. Analyses of gauge theories dimensionally reduced to matrix models have been useful in probing this duality. Noncommutative spaces also appear in string theories in certain backgrounds with a constant nonzero value for the 2-form gauge field [4, 5].

Fuzzy spaces are also closely related to the quantum Hall effect [10]. For the classic Landau problem of a charged particle in a magnetic field, the corresponding energy spectrum consists of equally spaced Landau levels; each Landau level is degenerate and the energy gap separating consecutive levels is proportional to the magnetic field B. For strong magnetic fields, the low-energy physics is confined to the states within one, say the lowest Landau level (LLL). The observables relevant for low energies are Hermitian operators on this subspace of the Hilbert space; they are given by the projection of the full operators to the lowest Landau level. The operators representing coordinates, for example, when projected to the LLL (or any other level), are no longer mutually commuting. The LLL thus becomes a model of the noncommutative 2-plane. (The appearance of noncommutativity in the string context mentioned above is similar, with the 2-form field playing the role of the magnetic field.) Generalizing beyond the plane, for the Landau problem on a compact space of finite volume, we get a finite number of states in the LLL, and the resulting subspace can be identified as $\mathcal{H}_N$, one of the ingredients for a fuzzy space. Observables then become $(N \times N)$-matrices and there are natural choices for the Laplacian. More specifically, the LLL states for quantum Hall effect on a space $\mathcal{M}$ give us a fuzzy version of $\mathcal{M}$.

The main advantage of this point of view is that the quantum Hall system gives us a model and a physical context to think about many issues related to fuzzy spaces. The lowest Landau level gives us a realization of the fuzzy space; subspaces, specified by a projection operator, will correspond to Hall droplets. The edge excitations of the Hall droplet describe the dynamics of the embedding of a disc into the fuzzy space. The bulk dynamics of the Hall droplet is related to the dynamics of gauge fields corresponding to isometries of the fuzzy space, and hence, gravity.

In the following, we will discuss such issues from both the matrix model–fuzzy space and the quantum Hall points of view. Thus, all results can have two different interpretations. We will start with the quantum Hall effect since this gives a familiar physical context.

2. Quantum Hall effect in higher dimensions

Quantum Hall effect in two dimensions is a very special physical phenomenon which has led to an enormous amount of theoretical and experimental research [11]. The basic phenomenon refers to the dynamics of charged fermions (electrons in a solid) in a plane with a constant magnetic field orthogonal to it. At the single particle level, the energy eigenstates are grouped into the Landau levels. For high values of the magnetic field at low temperatures, the separation of levels is high compared to the available thermal excitation energy and the dynamics is confined to the lowest Landau level. In a physical sample, there is also a potential V which confines the fermions to within the sample. If we have K fermions, they are localized near the minimum of V, but spread out over an area proportional to K due to the exclusion principle. We get an incompressible droplet. Physically interesting issues are the bulk dynamics of the droplet, which refers to its response to changes in the externally applied electromagnetic fields, and the edge dynamics which describes the fluctuations of the edge of the droplet. The electric current in the planar direction orthogonal to an applied in-plane electric field, the so-called Hall current, is quantized, hinting at topological robustness in the underlying dynamics. As a result, there are many interesting mathematical facets to the theory.

The quantum Hall effect was generalized to the four-dimensional sphere S^4 by Zhang and Hu [12]. Since then further generalizations and analyses in higher dimensions and different geometries have been carried out by many authors [13–20]. The general framework is the following. For any coset manifold of the G/H type, where G is a Lie group and H a compact subgroup (of dimension $\geqslant 1$), the spin connection gives the analogue of a constant background field. Thus, it is possible to consider QHE on such spaces taking the gauge field to be proportional to the spin connection. In two dimensions, one can consider $S^2 = SU(2)/U(1)$ which admits a constant $U(1)$ background field and leads to the usual QHE on a 2-sphere. For $S^4 = SO(5)/SO(4)$, the isotropy group is $H = SO(4) \sim SO(3) \times SO(3)$ giving the possibility of self-dual and anti-self-dual fields, the instantons. This was the case considered by Zhang and Hu [12]. For $\mathbf{CP}^k = SU(k+1)/U(k)$, one can get constant background fields which are either Abelian ($U(1)$) or non-Abelian ($U(k)$). Other interesting cases which have been studied include $S^3 = SU(2) \times SU(2)/SU(2)$ [17], the 8-sphere S^8 [18] and hyperbolic spaces based on noncompact groups [19].

The quintessential example for us is $\mathbf{CP}^k$, since it has all the characteristics we need and most of the other spaces which have been studied are special cases of this. The case of S^4 can be recovered from QHE on $\mathbf{CP}^3$ since $\mathbf{CP}^3$ is an S^2-bundle over S^4. As a result, $\mathbf{CP}^3$ with a $U(1)$ field leads to S^4 with a self-dual $SU(2)$ field as the background gauge field [14]. Likewise, since $\mathbf{CP}^7$ is a $\mathbf{CP}^3$-bundle over S^8, QHE on S^8 can be obtained from $\mathbf{CP}^7$ [18]. The case of S^3 can be related to $\mathbf{CP}^1 \times \mathbf{CP}^1 = S^2 \times S^2$ via the angle-axis embedding of $S^3/\mathbb{Z}_2$ in $S^2 \times S^2$ [17]. So, in short, we can use $\mathbf{CP}^k$ to formulate our calculations. Most of the results, of course, will be generic.

3. Quantum Hall effect on $\mathbf{CP}^k$

In this section, we shall consider the states in the lowest Landau level for the space $\mathbf{CP}^k = SU(k+1)/U(k)$ [13, 14]; this space will be adequate for our considerations.

The symmetries of $\mathbf{CP}^k$ form the group $SU(k+1)$, with $U(k)$ as the local isotropy group. The Riemannian curvature of $\mathbf{CP}^k$ takes values in the Lie algebra of $U(k)$, and because this is a homogeneous space, the curvature is constant in the basis of the frame fields. We can thus choose values of the background gauge field to be proportional to the curvature; this would give us a generalization of the 'constant magnetic field'. The Landau problem is defined by this choice of magnetic field and one can then solve for the Landau levels.

The construction of the wavefunctions for the Landau levels can be done as follows. Let g denote a general element of $SU(k+1)$ in the fundamental representation, i.e., it is a $(k+1)\times(k+1)$ matrix. The representative of g in a representation J is the Wigner $\mathcal{D}$-function corresponding to that representation. If $\hat{g}$ denotes a general operator version of g, then we may write the $\mathcal{D}$-function as

$$\mathcal{D}^{(J)}_{L,R}(g) = \langle J, l|\hat{g}|J, r\rangle, \tag{1}$$

where l, r label the states within the representation J. Functions on $SU(k+1)$ can be expanded in a basis of the $\mathcal{D}$-functions; functions on $\mathbf{CP}^k = SU(k+1)/U(k)$ are given by functions on $SU(k+1)$ which are $U(k)$-invariant.

We define the left and right translation operators on g by

$$L_A g = t_A g, \qquad R_A g = g t_A. \tag{2}$$

Here, t_A, $A = 1, 2, \ldots, k^2+2k$, are a set of Hermitian matrices which form a basis of the Lie algebra of $SU(k+1)$ in the fundamental representation. These are taken to obey

$$[t_A, t_B] = \mathrm{i} f_{ABC} t_C, \qquad \mathrm{Tr}(t_A t_B) = \tfrac{1}{2}\delta_{AB}. \tag{3}$$

f_{ABC} are the structure constants of $SU(k+1)$ in this basis. The right translation operators can be split into the subgroup and coset generators as follows. R_{k^2+2k} will denote the $U(1)$ generator in $U(k) \subset SU(k+1)$, R_a, $a = 1, 2, \ldots, k^2-1$, will denote $SU(k)$ generators. The coset components which are in the complement of $\underline{U(k)}$ in the Lie algebra $\underline{SU(k+1)}$ will be denoted by R_α, $\alpha = 1, 2, \ldots, 2k$. The coset generators can be further separated into the raising and lowering type $R_{\pm I} = R_{2I-1} \pm \mathrm{i} R_{2I}$, $I = 1, \ldots, k$. (A similar splitting can be made for the left translations, but they will not be needed for what follows.)

The translation operators R_A, L_A can be realized as differential operators with respect to the parameters of g. The coset operators R_α correspond to covariant derivatives while the $SU(k+1)$ operators L_A correspond to magnetic translations. In particular, the covariant derivatives on $\mathbf{CP}^k$ can be taken to be $D_{\pm I} = \mathrm{i}R_{\pm I}/R$, where R is a scale factor giving the radius of $\mathbf{CP}^k$. Since $[R_{\pm I}, R_{\pm J}] \in \underline{U(k)}$, we get $[R_{\pm I}, R_{\pm J}]f = 0$ for functions f on $\mathbf{CP}^k$ since they are $U(k)$-invariant. The commutator of the covariant derivatives on the wavefunctions of charged particles must be proportional to the field strength. Thus, they will not be true functions in $\mathbf{CP}^k$ but rather sections of a bundle. We consider a general background where there is a constant $U(1)$ field proportional to the $U(1)$ component of the curvature and a constant non-Abelian $SU(k)$ field proportional to the $SU(k)$ component of the curvature. The particles will be taken to have a unit Abelian charge and to transform as a representation J' of $SU(k)$ for the non-Abelian part. The statement about background fields can then be encoded in the commutation rules if we require the wavefunctions to obey

$$\begin{aligned} R_a \Psi_{m;a'} &= \Psi_{m;b'} (T_a)_{b'a'}, \\ R_{k^2+2k} \Psi_{m;a'} &= -\frac{nk}{\sqrt{2k(k+1)}} \Psi_{m;a'}. \end{aligned} \tag{4}$$

The indices $a', b' = 1, \ldots, N'$ label the states within the $SU(k)$ representation J'. The matrices T_a are the $SU(k)$ generators in the representation J'. For a unitary realization of the right translations, T_a should be the generators of a unitary representation of $SU(k) \in SU(k+1)$ and, for the $U(1)$ part, n has to be an integer, so that the $U(1)$ action is part of a unitary representation of $SU(k+1)$. These are Dirac-type quantization conditions. In the special case when there is no $SU(k)$ field, these simplify as

$$R_a \Psi_m = 0, \qquad R_{k^2+2k} \Psi_m = -\frac{nk}{\sqrt{2k(k+1)}} \Psi_m. \tag{5}$$

The wavefunctions obeying these conditions will be proportional to $\mathcal{D}^{(J)}_{L,R}$, where the state $|J, r\rangle$ is chosen to have the eigenvalue $-nk/\sqrt{2k(k+1)}$ for the $U(1)$ generator T_{k^2+2k} and to transform as the J' representation of $SU(k) \in SU(k+1)$. The representation J of $SU(k+1)$ must be so chosen that it contains such an $SU(k)$ representation, with the assigned $U(1)$ charge.

The Laplacian for the space is given by $-\nabla^2 = R_{+I} R_{-I} + R_{-I} R_{+I} = 2R_{+I} R_{-I} + \text{constant}$. The Hamiltonian for the Landau problem will be proportional to this for the nonrelativistic case and proportional to $\sqrt{-\nabla^2 + m^2}$ for the relativistic case; in any case, it is an increasing function of $R_{+I} R_{-I}$. We see that the minimum of the Hamiltonian, and hence the lowest Landau level, is given by wavefunctions obeying

$$R_{-I} \Psi_{m;a'} = 0. \tag{6}$$

This means that, for the lowest Landau level, in addition to the conditions (4), $|J, r\rangle$ must be a lowest weight state with $T_{-I}|J, r\rangle = 0$; we will denote these states as $|a', -n\rangle$. Once the representation J' is specified, one can identify representations J of $SU(k+1)$ which contain such a state. For example, if there is no $SU(k)$ field, the symmetric rank n representation of $SU(k+1)$ will contain the lowest weight state $|-n\rangle$, which is an $SU(k)$ singlet. The properly normalized wavefunctions are given by

$$\Psi_{m;a'}(g) = \sqrt{N} \langle J, l|\hat{g}|a', -n\rangle \equiv \sqrt{N} \mathcal{D}_{m;a'}(g), \tag{7}$$

where N is the dimension of the representation J of $SU(k+1)$. These are normalized by virtue of the orthogonality theorem

$$\int \mathrm{d}\mu(g) \mathcal{D}^*_{m;a'}(g) \mathcal{D}_{k;b'}(g) = \frac{\delta_{mk}\delta_{a'b'}}{N}. \tag{8}$$

It is instructive to relate this group-theoretic analysis to the standard discussion of $\mathbf{CP}^k$ in terms of homogeneous and local coordinates. We begin by recalling that $\mathbf{CP}^k$ is a $2k$-dimensional manifold parametrized by $k+1$ complex coordinates v_a, such that $\bar{v}_a v_a = 1$, with the identification $v_a \sim \mathrm{e}^{\mathrm{i}\theta} v_a$. One can further introduce local complex coordinates z_I, $I = 1, \ldots, k$, by writing

$$v_I = \frac{z_I}{\sqrt{1 + \bar{z} \cdot z}}, \qquad v_{k+1} = \frac{1}{\sqrt{1 + \bar{z} \cdot z}}. \tag{9}$$

We can now use a group element g in the fundamental representation of $SU(k+1)$ to parametrize $\mathbf{CP}^k$, by making the identification $g \sim gh$, where $h \in U(k)$. We can use the freedom of h transformations to write g as a function of the real coset coordinates x^i, $i = 1, \ldots, 2k$. The relation between the complex coordinates $z^I, \bar{z}^I$ in (9) and x^i is the usual one, $z^I = x^{2I-1} + \mathrm{i}x^{2I}$, $I = 1, \ldots, k$. The homogeneous coordinates are related to the group element by $g_{I,k+1} = v_I$, $g_{k+1,k+1} = v_{k+1}$.

For the variation of g, we can write

$$g^{-1}\,\mathrm{d}g = \left(-\mathrm{i}E_i^{k^2+2k} t_{k^2+2k} - \mathrm{i}E_i^a t_a - \mathrm{i}E_i^\alpha t_\alpha\right) \mathrm{d}x^i. \tag{10}$$

E_i^α are the frame fields in terms of which the Cartan–Killing metric on $\mathbf{CP}^k$ is given by

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = E_i^\alpha E_j^\alpha\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{11}$$

The Kähler 2-form on $\mathbf{CP}^k$ is likewise written as

$$\begin{aligned}\omega_K &= -\mathrm{i}\sqrt{\frac{2k}{k+1}}\,\mathrm{tr}(t_{k^2+2k}g^{-1}\,\mathrm{d}g\,g^{-1}\,\mathrm{d}g)\\ &= -\frac{1}{4}\sqrt{\frac{2k}{k+1}}\,f^{(k^2+2k)\alpha\beta}E_i^\alpha E_j^\beta\,\mathrm{d}x^i\wedge\mathrm{d}x^j \equiv \frac{1}{2}(\omega_K)_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}x^j.\end{aligned}\tag{12}$$

The fields $E_i^{k^2+2k}$ and E_i^a are related to the $U(1)$ and $SU(k)$ background gauge fields on $\mathbf{CP}^k$. In particular, the $U(1)$ field a is given by

$$a = \mathrm{i}n\sqrt{\frac{2k}{k+1}}\,\mathrm{tr}(t_{k^2+2k}g^{-1}\,\mathrm{d}g) = \frac{n}{2}\sqrt{\frac{2k}{k+1}}E^{k^2+2k}.\tag{13}$$

We can similarly define an $SU(k)$ background field $\bar{A}_i^a$. Its normalization is chosen so that

$$\bar{A}^a \equiv E^a = 2\mathrm{i}\,\mathrm{tr}(t^a g^{-1}\,\mathrm{d}g).\tag{14}$$

Note that $\bar{A}^a$ in (14) does not depend on n, while the Abelian field a in (13) is proportional to n. The corresponding $U(1)$ and $SU(k)$ background field strengths are

$$\begin{aligned}\partial_i a_j - \partial_j a_i &= n(\omega_K)_{ij} = -\frac{n}{2}\sqrt{\frac{2k}{k+1}}f^{(k^2+2k)\alpha\beta}E_i^\alpha E_j^\beta,\\ \bar{F}_{ij}^a &= \partial_i\bar{A}_j^a - \partial_j\bar{A}_i^a + f^{abc}\bar{A}_i^b\bar{A}_j^c = -f^{a\alpha\beta}E_i^\alpha E_j^\beta.\end{aligned}\tag{15}$$

We see from (15) that in the appropriate frame basis the background field strengths are constant, proportional to the $U(k)$ structure constants. It is in this sense that the field strengths in (15) correspond to uniform magnetic fields appropriate in defining QHE. The Maurer–Cartan equations

$$\mathrm{d}E^\alpha - \left(f^{a\alpha\beta}E^a + f^{(k^2+2k)\alpha\beta}E^{k^2+2k}\right)E^\beta = 0\tag{16}$$

show that the spin connections are given by $-f^{a\alpha\beta}E^a$ and $-f^{(k^2+2k)\alpha\beta}E^{k^2+2k}$; the field strengths (15) are thus proportional to the Riemann curvature of $\mathbf{CP}^k$.

The $U(1)$ background magnetic field (which leads to the Landau states) can be written in terms of the homogeneous coordinates as $a = -\mathrm{i}n\bar{v}\cdot\mathrm{d}v$ with the field strength

$$\mathrm{d}a = -\mathrm{i}n\,\mathrm{d}\bar{v}\cdot\mathrm{d}v = n\omega_K.\tag{17}$$

We can also write $n = 2BR^2$, in terms of the radius R of $\mathbf{CP}^k$, identifying B as the local value of the constant $U(1)$ magnetic field. (The case of charged fermions on $\mathbf{CP}^1 = S^2$ with $U(1)$ background field, corresponding to $k = 1$, was studied by Haldane several years ago [21]. In this case, the background gauge field a is that of a monopole of charge n placed at the origin of S^2.)

If there is only the $U(1)$ field, the representations of $SU(k+1)$ which are relevant for the lowest Landau level are totally symmetric and are of rank n. The wavefunctions are then explicitly given in local coordinates as

$$\begin{aligned}&\Psi_m(\vec{x}) = \sqrt{N}\left[\frac{n!}{i_1!i_2!\cdots i_k!(n-s)!}\right]^{\frac{1}{2}}\frac{z_1^{i_1}z_2^{i_2}\cdots z_k^{i_k}}{(1+\bar{z}\cdot z)^{n/2}}, \qquad m = 1,\ldots,N,\\ &s = i_1+i_2+\cdots+i_k, \qquad 0\leqslant i_i\leqslant n, \quad 0\leqslant s\leqslant n.\end{aligned}\tag{18}$$

These are the coherent states for $\mathbf{CP}^k$. The number of states in the lowest Landau level is given by the dimension of the symmetric rank n representation as

$$N = \dim J = \frac{(n+k)!}{n!k!}.\tag{19}$$

Note that, for large n, this gives $N \to n^k/k!$. When there is a non-Abelian background as well, the dimension N of the $SU(k+1)$ representation J depends on the particular J' representation chosen. While the full formula depends on the details of the representations, for large n we have

$$N = \dim J \to \dim J' \frac{n^k}{k!} = N' \frac{n^k}{k!}. \tag{20}$$

4. Matrix formulation of quantum Hall (phase space) dynamics

We are now in a position to present a matrix formulation of the dynamics of non-interacting fermions in the lowest Landau level, with and without external gauge interactions. Our analysis in this section will be quite general, not necessarily restricted to $\mathbf{CP}^k$.

We consider K fermions which occupy K states out of the N available states in the LLL. The confining potential $\hat{V}$ lifts the degeneracy of the LLL states and the fermions are localized around the minimum of $\hat{V}$ forming a droplet. Because of the exclusion principle and the conservation of the number of fermions, the excitations are deformations of the droplet which preserve the total volume of occupied states (volume of phase space).

The droplet is mathematically characterized by a diagonal density matrix $\hat{\rho}_0$ which is equal to 1 for occupied states and zero for unoccupied states. Further, $\hat{\rho}_0$ may be taken to be the density matrix for the many-body ground state. The most general fluctuations which preserve the LLL condition and the number of occupied states are unitary transformations of $\hat{\rho}_0$, namely $\hat{\rho}_0 \to \hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^\dagger$, where $\hat{U}$ is an $(N \times N)$ unitary matrix representing the dynamical modes. One can write an action for these modes as

$$S_0 = \int \mathrm{d}t\, \mathrm{Tr}[\mathrm{i}\hat{\rho}_0\hat{U}^\dagger\partial_t\hat{U} - \hat{\rho}_0\hat{U}^\dagger\hat{V}\hat{U}], \tag{21}$$

where $\hat{V}$ is the confining potential. (The Hamiltonian is $\hat{V}$ up to an additive constant.) The unitary matrix $\hat{U}$ can be thought of as a collective variable describing all the possible excitations within the LLL. The equation of motion resulting from (21) is the expected quantum Liouville equation for the density matrix $\hat{\rho}$,

$$\mathrm{i}\frac{\partial\hat{\rho}}{\partial t} = [\hat{V}, \hat{\rho}]. \tag{22}$$

The action S_0 can also be written as [14]

$$S_0 = \frac{N}{N'}\int \mathrm{d}\mu\, \mathrm{d}t\, \mathrm{tr}[\mathrm{i}(\rho_0 * U^\dagger * \partial_t U) - (\rho_0 * U^\dagger * V * U)], \tag{23}$$

where $\mathrm{d}\mu$ is the volume measure of the space where QHE has been defined and ρ_0, U, V are the symbols of the corresponding matrices on this space. (The hatted expressions correspond to matrices and unhatted ones to the corresponding symbols, which are fields on the space where QHE is defined.) Equation (23) is written for the case of non-Abelian fermions coupled to a background gauge field in some representation J' of dimension N'; the corresponding symbols are $(N' \times N')$ matrix valued functions. We will use 'Tr' to indicate the trace over the N-dimensional LLL Hilbert space while 'tr' indicates trace over the N'-dimensional representation J'. In the case of Abelian fermions, $N' = 1$ and tr is absent. (The large N limit we are considering will keep N' finite as $N \to \infty$.)

The general definitions of the symbol and the star product are as follows. If $\Psi_m(\vec{x}), m = 1, \ldots, N$, represent the correctly normalized LLL wavefunctions, then the symbol corresponding to an $(N \times N)$-matrix $\hat{O}$, with matrix elements O_{ml}, is

$$O(\vec{x}, t) = \frac{1}{N}\sum_{m,l} \Psi_m(\vec{x}) O_{ml}(t) \Psi_l^*(\vec{x}). \tag{24}$$

The star product is defined by the condition that the symbol for the product of two matrices is given as the star product of the symbols for the individual matrices, i.e., by $(\hat{O}_1\hat{O}_2)_{\text{symbol}} = O_1(\vec{x},t) * O_2(\vec{x},t)$.

Note that the dynamics of the underlying fermion problem is described in terms of a one-dimensional matrix action (21), which can also be written as a noncommutative field theory action, as in (23). The matrices and the action in (21) do not depend on the particular space and its dimensionality or the Abelian or non-Abelian nature of the underlying fermionic system. This information is encoded in equation (23) in the definition of the symbol, the star product and the measure.

This matrix formulation can be extended to include external gauge fields which are in addition to the uniform background magnetic field which defines the Landau problem [22]. These additional fields will be often referred to as the gauge field fluctuations. Gauge interactions should be described by a matrix action S which is invariant under time-dependent $U(N)$ rotations, $\hat{U} \to \hat{h}\hat{U}$, where $\hat{h} = \exp(-\mathrm{i}\hat{\lambda})$ for some Hermitian matrix $\hat{\lambda}$. The action will be the gauged version of S_0, with ∂_t replaced by the covariant derivative $\hat{D}_t = \partial_t + \mathrm{i}\hat{\mathcal{A}}$, where $\hat{\mathcal{A}}$ is a matrix gauge potential. Thus,

$$S = \int \mathrm{d}t\,\mathrm{Tr}[\mathrm{i}\hat{\rho}_0\hat{U}^\dagger(\partial_t + \mathrm{i}\hat{\mathcal{A}})\hat{U} - \hat{\rho}_0\hat{U}^\dagger\hat{V}\hat{U}]. \tag{25}$$

Invariance of this action under infinitesimal time-dependent $U(N)$ rotations $\delta\hat{U} = -\mathrm{i}\hat{\lambda}\hat{U}$ implies the following transformation for the gauge potential $\hat{\mathcal{A}}$:

$$\delta\hat{\mathcal{A}} = \partial_t\hat{\lambda} - \mathrm{i}[\hat{\lambda}, \hat{V} + \hat{\mathcal{A}}]. \tag{26}$$

The action (25) can be written in terms of the corresponding symbols as

$$S = \frac{N}{N'}\int \mathrm{d}t\,\mathrm{d}\mu\,\mathrm{tr}[\rho_0 * (\mathrm{i}U^\dagger * \partial_t U - U^\dagger * V * U - U^\dagger * \mathcal{A} * U)]. \tag{27}$$

The action (27) is now invariant under the infinitesimal transformations

$$\begin{aligned} &\delta U = -\mathrm{i}\lambda * U, \\ &\delta\mathcal{A} = \partial_t\lambda - \mathrm{i}(\lambda * (V + \mathcal{A}) - (V + \mathcal{A}) * \lambda). \end{aligned} \tag{28}$$

We shall refer to this as the W_N-gauge transformation, in analogy to the W_∞ transformation appearing in the case of the planar two-dimensional QHE [23, 24].

The key physical question is how the field $\mathcal{A}$ is related to the gauge fields A_μ to which the fermions couple in the usual way. Once this is known, the action (27) can be expressed in terms of the usual gauge fields. For the gauge interactions of the original fermion system, we have invariance under the usual gauge transformation

$$\delta A_\mu = \partial_\mu\Lambda + \mathrm{i}[\bar{A}_\mu + A_\mu, \Lambda], \qquad \delta\bar{A}_\mu = 0. \tag{29}$$

Here Λ is the infinitesimal gauge parameter and $\bar{A}_\mu$ is the non-Abelian uniform background field. What we need is an expression for $\mathcal{A}$ in terms of A_μ such that when the gauge fields A_μ are transformed as in (29), the field $\mathcal{A}$ undergoes the transformation (28). In other words, the transformation (28) is induced by the transformation (29). This is the basic principle which can be used to determine $\mathcal{A}$ as a function of A_μ, up to gauge-invariant terms. The bosonized action of the LLL fermionic system in the presence of gauge interactions then follows in a straightforward way. Since $\mathcal{A}$ is the time component of a noncommutative gauge field, the relation between $\mathcal{A}$ and the commutative gauge fields A_μ is essentially a Seiberg–Witten transformation [27, 28].

It is quite clear that the possible excitations of the LLL fermionic system are particle–hole excitations, which can, in principle, be described in terms of bosonic degrees of freedom.

The noncommutative field theories given by the actions S_0 in (23) and S in (27) are the exact bosonic actions describing the dynamics of the non-interacting LLL fermions without or with gauge interactions. Thus, the matrix theory provides a very general way to construct the bosonic action for a fermionic system by viewing it in phase space as a Landau problem with the symplectic structure being the magnetic field. Some of these ideas have already been used in the context of phase space bosonization for one-dimensional nonrelativistic fermions [24] and for the effective droplet dynamics in the planar quantum Hall effect [25, 26].

We shall now demonstrate that in the limit where $N \to \infty$ and the number of fermions is large, the action S_0 reduces, for arbitrary even dimensions, to a boundary action describing the edge excitations (Abelian and non-Abelian) of the QHE droplet [14]. In the presence of fluctuating gauge fields, there is an additional bulk action, given in terms of a Chern–Simons term, whose anomaly gets cancelled by the boundary contribution given in terms of a generalized chiral, gauged Wess–Zumino–Witten (WZW) action [22].

5. Star product for $\mathbf{CP}^k$ with $U(k)$ background gauge field

The large N simplifications are carried out using the symbols and star products. Let $\hat{X}$ be a general $(N \times N)$-matrix, with matrix elements X_{ml}, acting on the N-dimensional Hilbert space generated by the basis (7). The symbol corresponding to $\hat{X}$ is defined by

$$\begin{aligned} X_{a'b'}(\vec{x}, t) &= \frac{1}{N} \sum_{ml} \Psi_{m;a'}(\vec{x}) X_{ml}(t) \Psi^*_{l;b'}(\vec{x}) \\ &= \sum_{ml} \mathcal{D}_{m;a'}(g) X_{ml} \mathcal{D}^*_{l;b'}(g) = \langle b', -n | g^\dagger X^T g | a'; -n \rangle. \end{aligned} \tag{30}$$

In the non-Abelian case, the symbol is a $(N' \times N')$ matrix valued function, while in the Abelian case where J' is the singlet representation, the symbol is just a function on $\mathbf{CP}^k$. With this definition,

$$\operatorname{Tr} \hat{X} = \frac{N}{N'} \sum_{a'} \int \mathrm{d}\mu(g) X_{a'a'}(g). \tag{31}$$

The symbol corresponding to the product of two matrices $\hat{X}$ and $\hat{Y}$ is given by the star product of the symbols for $\hat{X}$ and $\hat{Y}$, i.e.,

$$\begin{aligned} (\hat{X}\hat{Y})_{a'b'} &= X_{a'c'} * Y_{c'b'} = \sum_{mrl} \mathcal{D}_{m;a'}(g) X_{mr} Y_{rl} \mathcal{D}^*_{l;b'}(g) \\ &= \langle b', -n | g^\dagger Y^T X^T g | a', -n \rangle = \langle b', -n | g^\dagger Y^T \mathbf{1} X^T g | a', -n \rangle. \end{aligned} \tag{32}$$

In order to evaluate the star product, we need to re-express the unit matrix $\mathbf{1}$ in (32), where $\mathbf{1} = \sum_m |m\rangle\langle m|$, and $|m\rangle$ are all the states in the J representation, in terms of the lowest weight states $|a', -n\rangle$. In the case of a $U(1)$ background field, the star product, following this method, was derived in [14]. We found

$$X * Y = \sum_s (-1)^s \left[\frac{(n-s)!}{n!s!} \right] \sum_{\sum i_k = s}^{n} \frac{s!}{i_1! i_2! \cdots i_k!} \big(R_{-1}^{i_1} \cdots R_{-k}^{i_k} X\big)\big(R_{+1}^{i_1} \cdots R_{+k}^{i_k} Y\big). \tag{33}$$

Expression (33) can be thought of as a series expansion in $1/n$. Similar expressions for the star product of functions were derived in the context of noncommutative $\mathbf{CP}^k$ [29].

In the case of the $U(k)$ background field, the calculation of the star product to arbitrary order in $1/n$ is very involved. The calculation to order $1/n$ was done in [14] and extended to order $1/n^2$ in [22]. The result is

$$X * Y = XY - \frac{1}{n} R_{-I} X R_{+I} Y + \frac{\mathrm{i}}{n^2} R_{-J} X f^{a\bar{I}J} (T_a)^T R_{+I} Y \\ + \frac{1}{2n^2} R_{-I} R_{-J} X R_{+I} R_{+J} Y + \mathrm{O}(1/n^3). \tag{34}$$

The right translation operators R_α can be expressed as differential operators using (2) and (10),

$$\begin{aligned} R_\alpha g &= \mathrm{i}(E^{-1})^i_\alpha \left(\partial_i g + \mathrm{i} g E_i^{k^2+2k} T_{k^2+2k} + \mathrm{i} g E_i^a T_a\right) \equiv \mathrm{i}(E^{-1})^i_\alpha D_i g, \\ R_\alpha g^\dagger &= \mathrm{i}(E^{-1})^i_\alpha \left(\partial_i g^\dagger - \mathrm{i} E_i^{k^2+2k} T_{k^2+2k} g^\dagger - \mathrm{i} E_i^a T_a g^\dagger\right) \equiv \mathrm{i}(E^{-1})^i_\alpha D_i g^\dagger, \end{aligned} \tag{35}$$

where Ts are the $U(k)$ generators in the particular representation g belongs to. Using (35) and the symbol definition (30), we find that the action of R_α on a symbol is

$$\begin{aligned} R_\alpha X_{a'b'} &= \mathrm{i}(E^{-1})^i_\alpha (D_i X)_{a'b'}, \\ D_i X &= \partial_i X + \mathrm{i}[\bar{A}_i, X], \qquad \bar{A}_i = \bar{A}_i^a (T_a)^T = E_i^a (T_a)^T, \end{aligned} \tag{36}$$

where $\bar{A}$ is the $SU(k)$ background gauge field in the J' representation. Note that the $U(1)$ part of the gauge field does not contribute in (36).

Combining expressions (34) and (36), we can rewrite the star product in terms of covariant derivatives and real coordinates (instead of complex) as

$$X * Y = XY + \frac{1}{n} P^{ij} D_i X D_j Y - \frac{\mathrm{i}}{n^2} P^{il} P^{kj} D_i X \bar{F}_{lk} D_j Y \\ + \frac{1}{2n^2} P^{ik} P^{jl} \mathcal{D}_i D_j X \mathcal{D}_k D_l Y + \mathrm{O}(1/n^3), \tag{37}$$

where $\bar{F}_{lk} = \bar{F}^a_{lk} (T_a)^T$ and $P^{ij} = g^{ij} + \frac{\mathrm{i}}{2}\left(\omega_K^{-1}\right)^{ij}$. $\mathcal{D}_i$ is the Levi-Civita covariant derivative for a curved space such as $\mathbf{CP}^k$, namely

$$\begin{aligned} \mathcal{D}_i D_j X &\equiv D_i D_j X - \Gamma^l_{ij} D_l X, \\ D_i E^\alpha_j &= \partial_i E^\alpha_j + f^{\alpha A \beta} E^A_i E^\beta_j = \Gamma^l_{ij} E^\alpha_l, \end{aligned} \tag{38}$$

where A in $f^{\alpha A\beta}$ is a $U(k)$ index (both $U(1)$ and $SU(k)$) and Γ^l_{ij} is the Christoffel symbol for $\mathbf{CP}^k$.

Equation (37) is valid for both Abelian and non-Abelian cases. In the Abelian case, they simplify since X, Y are commuting functions and $\bar{F}_{lk} \to 0$, so that $D_i X \to \partial_i X$ and $\mathcal{D}_i D_j X \to \partial_i \partial_j X - \Gamma^l_{ij} \partial_l X$.

6. Calculation of $\mathcal{A}$

In this section, we will outline the calculation of $\mathcal{A}$ as a function of A_μ via the implementation of the W_N-transformation (28) as induced by the gauge transformation (29) on A_μ. Using (28) and (37), we find that up to $1/n^2$ terms

$$\delta\mathcal{A} = \partial_t \lambda - \mathrm{i}[\lambda, V + \mathcal{A}] - \frac{\mathrm{i}}{n} P^{ij} (D_i \lambda D_j (V + \mathcal{A}) - D_i (V + \mathcal{A}) D_j \lambda) \\ - \frac{1}{n^2} P^{il} P^{kj} (D_i \lambda \bar{F}_{lk} D_j V - D_i V \bar{F}_{lk} D_j \lambda) \\ - \frac{\mathrm{i}}{2n^2} P^{ik} P^{jl} (\mathcal{D}_i D_j \lambda \mathcal{D}_k D_l V - \mathcal{D}_i D_j V \mathcal{D}_k D_l \lambda). \tag{39}$$

At this stage, it is useful to discuss the scaling of various quantities. All expressions so far (including the measure $\mathrm{d}\mu$, g^{ij}, $\left(\omega_K^{-1}\right)^{ij}$, etc) have been written in terms of the dimensionless coordinates $x_i = \tilde{x}_i/R$, where R is the radius of $\mathbf{CP}^k$ and $\tilde{x}$ are the dimensionful coordinates. The calculation of the star product (37) involves a series expansion in terms of $1/n$, where $n = 2BR^2$ and B is the magnitude of the constant $U(1)$ magnetic field. Written in terms of the dimensionful parameters $\tilde{x}_i$, the expansion in $1/n$ becomes an expansion in $1/B$. We shall further assume that the energy scale of the gauge field fluctuation A_μ, and therefore of $\mathcal{A}$, is much smaller than B to be consistent with the restriction to LLL.

The scale of the confining potential V is set by the magnetic field B ($\sim n$ in terms of dimensionless variables). A convenient choice for the confining matrix potential $\hat{V}$ is such that the ground-state density $\rho_0(\vec{x})$ corresponds to a spherical droplet. This is the case when all the $SU(k)$ multiplets of the J representation up to a fixed hypercharge (the eigenvalue of T_{k^2+2k}) are completely filled, starting from the lowest. A simple choice for such a potential is that used in [14],

$$\hat{V} = \sqrt{\frac{2k}{k+1}}\,\nu\left(T_{k^2+2k} + \frac{nk}{\sqrt{2k(k+1)}}\right), \tag{40}$$

where ν is a constant. (The potential does not have to be exactly of this form; any potential with the same qualitative features will do.) The particular expression (40) has the property that $\langle s|\hat{V}|s\rangle = \nu s$, where $|s\rangle$ denotes an $SU(k)$ multiplet of hypercharge $-nk+sk+s$, namely $\sqrt{2k(k+1)}T_{k^2+2k}|s\rangle = (-nk+sk+s)|s\rangle$. The symbol for (40) was calculated in [14] to be

$$V_{a'b'} \equiv \langle b', -n|g^\dagger V^T g|a', -n\rangle = \nu n\frac{\bar{z}\cdot z}{1+\bar{z}\cdot z}\delta_{a'b'} + S_{k^2+2k,a}(T_a)_{b'a'}, \tag{41}$$

where $S_{k^2+2k,a} = 2\,\mathrm{tr}(g^\dagger t_{k^2+2k} g t_a)$. The important point is that the first term in (41) is diagonal and of order n in terms of the dimensionless variables z, while the second non-diagonal term is of order n^0. In analysing (39), we can absorb the order n^0 term of the confining potential in the definition of $\mathcal{A}$ and treat separately the diagonal term of order n as the potential V to be used for large n simplifications. In this way, since $V_{a'b'} = \delta_{a'b'}V(r)$, where $r^2 = \bar{z}\cdot z$, is proportional to the identity, expression (39) can be further simplified as

$$\begin{aligned}\delta\mathcal{A} = \partial_t\lambda - \mathrm{i}[\lambda,\mathcal{A}] - \frac{\mathrm{i}}{n}P^{ij}(D_i\lambda D_j\mathcal{A} - D_i\mathcal{A}D_j\lambda)\\ + u^i D_i\lambda - \frac{\mathrm{i}}{n}(P^{il}D_i\lambda\bar{F}_{lk} - P^{li}\bar{F}_{lk}D_i\lambda)u^k\\ + \frac{1}{2n^2}\left[\left(\omega_K^{-1}\right)^{ik}g^{jl} + g^{ik}\left(\omega_K^{-1}\right)^{jl}\right]\mathcal{D}_i D_j\lambda\nabla_k\partial_l V,\end{aligned} \tag{42}$$

where

$$\nabla_k\partial_l V = \partial_k\partial_l V - \Gamma^n_{kl}\partial_n V, \qquad u^i = \frac{1}{n}\left(\omega_K^{-1}\right)^{ij}\partial_j V. \tag{43}$$

The quantity u^i is essentially the phase space velocity, if we think of the LLL as the phase space of a lower dimensional system, with symplectic structure $n\omega_K$ and Hamiltonian V.

Equation (42) gives the transformation of $\mathcal{A}$. What we are seeking is an expression for $\mathcal{A}$ as a function of A_μ, $\mathcal{A} = f(A_\mu)$, such that

$$\begin{aligned}&\delta\mathcal{A}(\text{as in equation (42)}) = f(\delta A_\mu),\\ &\delta A_\mu = \partial_\mu\Lambda + \mathrm{i}[\bar{A}_\mu + A_\mu, \Lambda] = D_\mu\Lambda + \mathrm{i}[A_\mu,\Lambda], \qquad \delta\bar{A}_\mu = 0.\end{aligned} \tag{44}$$

The solution for $\mathcal{A}$ can be worked out from this requirement, although the calculation is algebraically a bit tedious [22]. It is given by

$$
\begin{aligned}
\mathcal{A} = A_0 &- \frac{\mathrm{i}}{2n} g^{ij}[A_i, 2D_i A_0 - \partial_0 A_i + \mathrm{i}[A_i, A_0]] + \frac{1}{4n}\left(\omega_K^{-1}\right)^{ij}\{A_i, 2D_j A_0 - \partial_0 A_j + \mathrm{i}[A_j, A_0]\} \\
&+ u^i A_i - \frac{\mathrm{i}}{2n} g^{ij}[A_i, A_k]\partial_j u^k + \frac{1}{4n}\left(\omega_K^{-1}\right)^{ij}\{A_i, A_k\}\partial_j u^k \\
&- \frac{\mathrm{i}}{2n} g^{ij}[A_i, 2D_j A_k - D_k A_j + \mathrm{i}[A_j, A_k] + 2\bar{F}_{jk}]u^k \\
&+ \frac{1}{4n}\left(\omega_K^{-1}\right)^{ij}\{A_i, 2D_j A_k - D_k A_j + \mathrm{i}[A_j, A_k] + 2\bar{F}_{jk}\}u^k \\
&+ \frac{1}{2n^2} g^{ik}\left(\omega_K^{-1}\right)^{jl}(\mathcal{D}_i A_j + \mathcal{D}_j A_i)\nabla_k \partial_l V,
\end{aligned}
\tag{45}
$$

where [,] and {,} indicate commutators and anticommutators, respectively. The symbol for the matrix gauge transformation parameter $\hat{\lambda}$ can also be evaluated as

$$
\lambda = \Lambda - \frac{\mathrm{i}}{2n} g^{ij}[A_i, D_j \Lambda] + \frac{1}{4n}\left(\omega_K^{-1}\right)^{ij}\{A_i, 2D_j \Lambda\} + \mathrm{O}(1/n^2). \tag{46}
$$

The gauge field A_μ in (45) contains both the Abelian $U(1)$ and non-Abelian $SU(k)$ components. In the Abelian case where the fermions interact only with the $U(1)$ gauge field, the symbols are commuting functions, so the commutator terms in (45) vanish. In terms of the dimensionful quantities $\tilde{x} = Rx$, $\tilde{D} = D/R$, $\tilde{A} = A/R$, $\tilde{V} \sim B$, $\mathcal{A}$ can be written as a series expansion in $1/B$. The terms shown in (45) account for all terms of order B^0 and $1/B$.

The function $\mathcal{A}$ being the symbol of the time component of the matrix gauge potential can be thought of as the Seiberg–Witten map [27, 28] for a curved manifold in the presence of non-Abelian background gauge fields.

It is clear from (44) that expression (45) is only determined up to gauge-invariant terms whose coefficients are not constrained by the W_N-transformation (28) and the requirement that it is induced via the gauge transformation (29). As we shall see in the next section, this solution produces the minimal gauge coupling for the chiral field describing the edge excitations of the quantum Hall droplet.

7. Edge and bulk actions and anomaly cancellation

The simplification of the action (27) requires one more ingredient, namely the symbol for $\hat{\rho}_0$. For the case of a confining potential $\hat{V}$ with an $SU(k)$ symmetry, as discussed in the previous section, one can perform an exact calculation for ρ_0. In the limit where N is large and the number of fermions K is large, where $N \gg K$, one can show that the symbol corresponding to the density matrix is of the form

$$
(\rho_0)_{a'b'} = \rho_0(r^2)\delta_{a'b'}, \qquad \rho_0(r^2) = \Theta\left(1 - \frac{R^2 r^2}{R_{\mathrm{D}}^2}\right), \tag{47}
$$

where Θ is the step function and R_{D} is the radius of the droplet. Equation (47) defines the so-called droplet approximation for the fermionic density. We want to evaluate the action S, and identify the edge and bulk effective actions, in this approximation. As mentioned earlier, the $1/n$ expansion of various quantities can be thought of as an expansion in $1/B$ if we write our expressions in terms of the dimensionful coordinates $\tilde{x}$. Similarly, using (20), the prefactor $(N/N')\,\mathrm{d}\mu \rightarrow [n^k/(k!R^{2k})]\,\mathrm{d}\tilde{\mu} = (2B)^k/k!\,\mathrm{d}\tilde{\mu}$, where $\mathrm{d}\tilde{\mu}$ is the measure of the space in terms of the dimensionful coordinates. For convenience, we will continue the evaluation of the edge and bulk effective actions in terms of the dimensionless coordinates, keeping in mind, though, that the $1/n$ expansion can always be converted to a $1/B$ expansion with the appropriate overall prefactor to correctly accommodate the volume of the droplet.

The large n limit of the bosonic action S_0 in the absence of gauge interactions was derived in [14]. It can be written in terms of a unitary $(N' \times N')$ matrix valued field $G = \mathrm{e}^{\mathrm{i}\Phi}$, where Φ is the symbol corresponding to $\hat{\Phi}$ in $\hat{U} = \mathrm{e}^{\mathrm{i}\hat{\Phi}}$. Explicitly,

$$S_0 = -\frac{N}{2nN'}\int \mathrm{d}t\,\mathrm{d}\mu \frac{\partial \rho_0}{\partial r^2}\,\mathrm{tr}[(G^\dagger \dot{G} + \nu G^\dagger D_\omega G)G^\dagger D_\omega G] \\ + \frac{Nk}{4\pi n N'}\int \rho_0 \left[-\mathrm{d}(\mathrm{i}\bar{A}\,\mathrm{d}G G^\dagger + \mathrm{i}\bar{A}G^\dagger\,\mathrm{d}G) + \frac{1}{3}(G^\dagger\,\mathrm{d}G)^3\right] \wedge \left(\frac{\omega_K}{2\pi}\right)^{k-1}. \quad (48)$$

In this equation, D_ω is the component of the covariant derivative D perpendicular to the radial direction, along a special tangential direction on the droplet boundary, given explicitly as $D_\omega = -\left(\omega_K^{-1}\right)^{ij} 2r\hat{x}_i D_j$; $\hat{x}_i$ is the radial unit vector normal to the boundary of the droplet. ν is the parameter displayed in (40) for that particular potential. For a more general potential,

$$\nu = \frac{1}{n}\frac{\partial V}{\partial r^2}\bigg]_{\text{boundary}}. \quad (49)$$

The volume element $\mathrm{d}\mu$ for $\mathbf{CP}^k$ is normalized such that $\int \mathrm{d}\mu = 1$ and is given in local coordinates as

$$\mathrm{d}\mu = \epsilon^{i_1 j_1 i_2 j_2 \cdots i_k j_k}(\omega_K)_{i_1 j_1}\cdots(\omega_K)_{i_k j_k}\frac{\mathrm{d}^{2k}x}{(4\pi)^k} = k!\sqrt{\det \omega_K}\,\frac{\mathrm{d}^{2k}x}{(2\pi)^k}. \quad (50)$$

Since $\rho_0(r^2)$ is a step function as in (47), its derivative $\partial\rho_0/\partial r^2$ produces a delta function with support at the boundary of the droplet. As a result, the first two terms in (48) are boundary terms. The action S_0 in (48) is a higher dimensional generalization of a chiral, Wess–Zumino–Witten action, vectorially gauged with respect to the time-independent background gauge field $\bar{A}$ [30]. The third term is a WZW term written as an integral over a $(2k+1)$-dimensional region, corresponding to the droplet and time. The usual 3-form in the integrand of the WZW term, $(G^\dagger \mathrm{d}G)^3$, has now been augmented to the appropriate $(2k+1)$-form $(G^\dagger\,\mathrm{d}G)^3 \wedge (\omega_K)^{k-1}$. Since the WZW term is the integral of a locally exact form, the whole action S_0 should be considered as part of the edge action.

The part of the action which depends on the external gauge field A_μ is given by

$$S_A = -\frac{N}{N'}\int \mathrm{d}t\,\mathrm{d}\mu\,\mathrm{tr}[\rho_0 * U^\dagger * \mathcal{A} * U]. \quad (51)$$

The large n limit of S_A was evaluated in [22]. It contains a boundary contribution expressing the interaction between the matter field G characterizing the edge excitations of the quantum Hall droplet and the external gauge field A_μ and a bulk contribution written solely in terms of the gauge field fluctuations A_μ. Combining the large n limits for S_0 and S_A, the total edge action is essentially a higher dimensional Wess–Zumino–Witten action chirally gauged with respect to the external gauge field A_μ (up to gauge-invariant, completely A-dependent terms). The bulk contribution is written in the form of Chern–Simons actions (when the fields A_μ are slowly varying with respect to the length scale set by B).

To keep the expressions simple, we first write the results when $u^i = 0$, i.e., $\omega_K \rfloor \mathrm{d}V = 0$. The edge action is then

$$S^{\text{edge}}(u^i = 0) = \frac{N}{2nN'}\bigg[\int \partial_i\rho_0\left(\omega_K^{-1}\right)^{ij} G^\dagger\left(\partial_0 + \mathrm{i}A_0^L G - \mathrm{i}GA_0^R\right)G^\dagger\left(\partial_j G + \mathrm{i}A_j^L G - \mathrm{i}GA_j^R\right) \\ + \frac{k}{2\pi}\int \rho_0\bigg[-\mathrm{d}(\mathrm{i}A^L\,\mathrm{d}G\,G^\dagger + \mathrm{i}A^R G^\dagger\,\mathrm{d}G + A^L G A^R G^\dagger) \\ + \frac{1}{3}(G^\dagger\,\mathrm{d}G)^3\bigg] \wedge \left(\frac{\omega_K}{2\pi}\right)^{k-1}\bigg] = S_{\text{WZW}}(A^L = A + \bar{A},\, A^R = \bar{A}). \quad (52)$$

This is clearly a higher dimensional WZW action, gauged in a left–right asymmetric way. The full S^{edge} action, including the u^i dependent terms, is also a gauged WZW action; it is obtained from (52) by the following substitutions:

$$\begin{aligned}
&\partial_0 \to \partial_\tau = \partial_0 + u^k \partial_k, \\
&A_0^L \to A_\tau^L = A_0 + u^k(A_k + \bar{A}_k), \qquad A_0^R \to A_\tau^R = u^k \bar{A}_k, \\
&A_i^L = A_i + \bar{A}_i, \qquad A_i^R = \bar{A}_i.
\end{aligned} \tag{53}$$

The derivative ∂_τ is the convective derivative, well known in hydrodynamics. The appearance of A_τ is consistent with the gauging of the convective derivative. One can explicitly verify that the u-dependent terms generated by (53) from (52) are gauge invariant.

Because of the chiral gauging, S^{edge} (including the u^i-dependent terms) is not gauge invariant. Under a gauge transformation, it changes by

$$\delta S^{\text{edge}} = \frac{Nk}{4\pi n N'} \int \mathrm{d}\rho_0 \,\mathrm{tr}[\mathrm{d}(A + \bar{A})\Lambda] \wedge \left(\frac{\omega_K}{2\pi}\right)^{k-1}. \tag{54}$$

The bulk contribution to the action is given by

$$\begin{aligned}
S^{\text{bulk}} = &-\frac{N}{N'} \int \mathrm{d}t\, \mathrm{d}\mu\, \rho_0 \,\mathrm{tr}(A_0 + u^k A_k) \\
&+ \frac{kN}{4\pi n N'} \int \mathrm{d}t\, \rho_0 \left[\mathrm{tr}\left((A + \bar{A})\,\mathrm{d}(A + \bar{A}) + \frac{2\mathrm{i}}{3}(A + \bar{A})^3\right) \wedge \left(\frac{\omega_K}{2\pi}\right)^{k-1} \right.\\
&\left. - \frac{(k-1)}{2\pi}\,\mathrm{tr}\left[\left((A + \bar{A})\,\mathrm{d}(A + \bar{A}) + \frac{2\mathrm{i}}{3}(A + \bar{A})^3\right)\mathrm{d}V\right] \wedge \left(\frac{\omega_K}{2\pi}\right)^{k-2}\right] \\
&+ \frac{N}{2nN'} \int \mathrm{d}t\, \mathrm{d}\mu\, \rho_0 \,\mathrm{tr}[\nabla^i F_{ik} + (k+1)A_k]u^k.
\end{aligned} \tag{55}$$

The metric-dependent terms in the last line of (55) can be neglected compared to the rest of the terms. Written in terms of the dimensionful coordinates $\tilde{x}$, they get a prefactor proportional to $1/(BR^2)$; they are small compared to the other terms in the approximation where R is large and the gradients of the external field are small compared to B. (The actions (52) and (55) are related to the Kähler–Chern–Simons and Kähler–WZW actions [31].)

The V-dependent terms in S^{bulk} can be shown to be gauge invariant for a spherically symmetric ρ_0 and V. The lack of gauge invariance is entirely due to the Kähler–Chern–Simons term in the second line of (55). The change in S^{bulk} under a gauge transformation is

$$\delta S^{\text{bulk}} = -\frac{Nk}{4\pi n N'} \int \mathrm{d}\rho_0 \,\mathrm{tr}[\mathrm{d}(A + \bar{A})\Lambda] \wedge \left(\frac{\omega_K}{2\pi}\right)^{k-1}. \tag{56}$$

Adding the gauge variations of the edge and bulk actions we find, as expected, that the total bosonic action S is gauge invariant, $\delta S = \delta S^{\text{edge}} + \delta S^{\text{bulk}} = 0$. (Anomaly cancellation for two-dimensional Hall effect is discussed in [32].)

The phenomenon of anomaly cancellation is of course expected since gauge invariance is already built in the action S. The full bosonic action S is, by construction, invariant under

$$\delta U = -\mathrm{i}\lambda * U, \qquad \delta A_\mu = D_\mu \Lambda + \mathrm{i}[A_\mu, \Lambda] \tag{57}$$

via the induced W_N-transformation (28). This also implies the following gauge transformation for G:

$$\delta G G^\dagger = -\mathrm{i}\Lambda + \cdots, \tag{58}$$

where the ellipsis indicates terms of higher order in $1/n$. This means that the large n limit of the effective action $S = S^{\text{edge}} + S_A^{\text{bulk}}$ is automatically gauge invariant under (57) and (58), guaranteeing the anomaly cancellation between the edge and bulk contributions.

In the next section, we will outline a different derivation of the bulk action by matrix techniques. We shall see that the 'topological' part of (55), where the last two terms are neglected, can be written, for $\rho_0 = \mathbf{1}$, as a single $(2k+1)$-dimensional Chern–Simons term to all orders in $1/n$. In fact, with a little bit of algebra, and using $N/N' = n^k/k!$ for large n, the bulk action (55) can be brought to the form

$$S_A^{\text{bulk}} = S_{\text{CS}}(\tilde{A}), \qquad \tilde{A} = (A_0 + V, -a_i + \bar{A}_i + A_i). \tag{59}$$

The gauge fields in this equation are of the form $A^a\left(T_a^T\right)$. Since only the last eigenvalue of the generator T_{k^2+2k} contributes in the symbol, we may write the Abelian part as $-a_i = -na_{Ki} = a_{Ki}T^T_{k^2+2k}\sqrt{\frac{2(k+1)}{k}}$. The combination $-a_i + \bar{A}_i + A_i$ can therefore be written as $a + A$, where all components are expanded using T^T, a being the full background field $\bar{A}^{k^2+2k}T^T_{k^2+2k} + \bar{A}^a T^T_a$. Note that the anti-Hermitian components are $\mathrm{i}A^A T^T_A = -\mathrm{i}A^A\left(-T^T_A\right) = -\mathrm{i}A^A(T_A)_{\bar{R}}$, where the index A denotes both the $U(1)$ and $SU(k)$ indices. The matrices $-T^T_A$ are the generators in the representation $\bar{R}$ conjugate to the representation R of T_A. We will use this in the next section.

7.1. The nature of the edge states

Turning now to the nature of the edge states, this can be understood in terms of the field G in (48) [14]. First, consider the case of the background field being Abelian, so that one can write $G = \mathrm{e}^{\mathrm{i}\Phi}$, where Φ is just one function, not a matrix. The surface of the droplet is topologically S^{2k-1}. The action involves time derivatives of Φ and D_ω which is the derivative along an angular direction on S^{2k-1} which is ω_K conjugate to the radius of the droplet. It is convenient to decompose Φ in terms of the eigenstates of D_ω. Since $S^{2k-1}/S^1 = \mathbf{CP}^{k-1}$, the surface of the droplet, other than the angular direction corresponding to D_ω, will be $\mathbf{CP}^{k-1}$. We see that for each eigenvalue of D_ω, Φ can be expanded in terms of the $\mathcal{D}$-functions for $SU(k)$, with the eigenvalue for the right action of R_{k^2-1} fixed to the eigenvlaue for D_ω. Explicitly, we can write $\Phi = \sum_l \sum_{p,q|p-q=s} c_m^{p,q}\mathcal{D}^{(p,q)}_{m;s}(h)$, where $h \in SU(k)$, $\mathcal{D}^{(p,q)}_{m;s}(h)$ is the $\mathcal{D}$-function for the irreducible representation of $SU(k)$ of the tensorial type T^q_p with p symmetric lower indices, q symmetric upper indices and the contraction (or trace) of any p-type index with any q-type index must vanish. The eigenvalue of D_ω is $s = p - q$, up to normalization factors. The right state $|s\rangle$ in the $\mathcal{D}$-function denotes the unique $SU(k-1)$-invariant state for this representation, with the given eigenvalue for R_{k^2-1}. Note that $\mathcal{D}^{(p,q)}_{m;s}$ are similar to wavefunctions of a reduced Landau problem on $\mathbf{CP}^{k-1}$. More details of edge states can be found in [12, 16, 20]. The analysis of edge states on S^4 starting from $\mathbf{CP}^3$ can be found in [14]. For the case of a non-Abelian background, one can carry out a similar analysis, although the details are more involved.

8. The fuzzy space point of view

We shall now return to the question of taking the large N limit of a matrix action, focusing on the fuzzy space–matrix model point of view. The basic strategy has been to introduce a set of wavefunctions for the Hilbert space $\mathcal{H}_N$ and then the large N limit can be defined using the symbols for the matrices involved. But, as mentioned in the introduction, there are many ways to do this. Since the action we start with is a matrix action, there is, initially, no notion of space or spatial geometry. The Hilbert space $\mathcal{H}_N$ on which the matrices act as linear transformations can be taken, for example, as arising from the quantization of the phase space $S^2 = SU(2)/U(1)$, where the symplectic form is $\omega = -\mathrm{i}n\omega_K$, ω_K being the Kähler form on S^2. (N will be a function of n.) Taking n large in this way defines a specific large N limit. Since ω is a background $U(1)$ field on S^2, we could also consider a deformation of this situation with, say, $\omega = -\mathrm{i}n\omega_K + F$, where F is topologically trivial (so that the dimension of $\mathcal{H}_N$ is not changed). The wavefunctions to be used would now be modified and the large n limit, via the modified symbols, gives S^2 with a different choice of background field on it. One could also consider $\mathcal{H}_N$ as the quantization of, say, $\mathbf{CP}^k = SU(k+1)/U(k)$, with a suitable choice of symplectic form (with the values of N matching the dimensions of a class of $SU(k+1)$ irreducible representations). This is what we did in the analysis for the dynamics of the quantum Hall droplet. It is clear that there are many ways to take the large N limit. Even for the same geometry and topology for the phase space, the choice of symplectic form is not unique. Once the dimension of the symplectic space has been chosen, these limits can be parametrized by the choice of background gauge fields. This is what we want to analyse, particularly for the matrix action $S = \mathrm{i}\int \mathrm{d}t\, \mathrm{Tr}\rho_0(U^\dagger D_0 U)$, where $D_0 = \partial_0 + A_0$. (We will use anti-Hermitian A_0 for simplicity of notation in this section. Also any potential V can be included in A_0.)[3]

The result we find will be essentially identical, with some reinterpretation, to the result for the quantum Hall system. However, keeping in mind the fuzzy geometry, we want to take the point of view that the Hilbert space is the fundamental entity, with the smooth manifold being just a large n simplification. It is, therefore, important to have a matrix version of the calculations for extracting the large n expansion.

The unitary transformation U encodes the fluctuations of the chosen density matrix or the edge states from the quantum Hall point of view. Equivalently, it gives the boundary effects for dynamics in a subspace of a fuzzy space. The bulk dynamics is not sensitive to U and can be extracted by taking $\rho_0 = \mathbf{1}$. Effectively, we are then seeking the simplification of $S = \mathrm{i}\int \mathrm{d}t\, \mathrm{Tr}\, D_0$ in the limit of large matrices. This action is the one-dimensional Chern–Simons action for the matrix theory.

In the following, we shall choose a specific background and expand the action around it. The final result is not sensitive to the details of the background, except for the dimension and topology. Therefore, we can choose a simple background, say, $\mathbf{CP}^k$ with only the Abelian field; thus $\omega = -\mathrm{i}n\omega_K$. The gauge fields A_0, A_i, which can be Abelian or non-Abelian, will be expanded around this background; thus, the fields A_0, A_i are actually functions on fuzzy $\mathbf{CP}^k$.

At this point, it is appropriate to clarify the relationship between the lowest Landau level and fuzzy geometry in more specific terms. The states of the lowest Landau level form an N-dimensional Hilbert space which we identify as the space $\mathcal{H}_N$ needed for fuzzy $\mathbf{CP}^k$. Observables when restricted to the LLL are $(N\times N)$-matrices and these can be taken as functions on fuzzy $\mathbf{CP}^k$. We can see that these are in correspondence with functions on smooth $\mathbf{CP}^k$. A basis for functions on smooth $\mathbf{CP}^k$ is of the form $\left\{\mathcal{D}^R_{m,w}(g)\right\}$ where $|w\rangle$ is trivial under the action of $U(k) \subset SU(k+1)$, so that we get true functions on $SU(k+1)/U(k)$ and R is any representation which contains such a state. At the matrix level, since the states are symmetric representations of $SU(k+1)$, a general matrix is of the form $X^{b_1 b_2\cdots b_n}_{a_1 a_2\cdots a_n}$ and transforms as the product representation $\bar{J}\otimes J$. The reduction of this product will contain the singlet, the adjoint and higher irreducible representations. Thus, upon reducing the product $\bar{J}\otimes J$, we can write a matrix X in terms of a basis corresponding to the irreducible representations of $SU(k+1)$ as

$$X^{b_1 b_2\cdots b_n}_{a_1 a_2\cdots a_n} = \sum_{0\leqslant p\leqslant n} \sum_{\{A\}} C^{A_1 A_2\cdots A_p} \left(T_{A_1 A_2\cdots A_p}\right)^{b_1 b_2\cdots b_n}_{a_1 a_2\cdots a_n}. \tag{60}$$

[3] We do not use a hat to represent matrices or operators on $\mathcal{H}$ from now on to avoid clutter in notation. Whether we mean the matrix or the symbol should be clear from the context. A_0 now plays the role of $\hat{\mathcal{A}}$ of section 4.

The matrices $T_{A_1A_2\cdots A_p}$ are obtained from products of the generators of $SU(k+1)$, namely T_As, with the condition that they are traceless for any contraction of any of the a_is with any of the b_js. They form a complete basis at the matrix level. The symbol corresponding to the identity is the constant function on smooth $\mathbf{CP}^k$, the symbol for T_A will be of the form $\mathcal{D}^{(\text{adj})}_{A,w}(g)$. The symbols corresponding to $T_{A_1A_2\cdots A_p}$ are $\mathcal{D}^{R}_{m,w}(g)$, for the appropriate representation R. We see that the symbol corresponding to X is a function on $\mathbf{CP}^k$, expandable in terms of a truncated set of basis functions since $p \leqslant n$. As $n \to \infty$, 'functions' on $\mathcal{H}_N$ tend to functions on $\mathbf{CP}^k$. Further, the star product shows that the algebra of Mat_N goes over to the commutative algebra of functions on $\mathbf{CP}^k$.

This argument is for wavefunctions of the LLL corresponding to an Abelian background. The wavefunctions for the LLL with a non-Abelian background field are of the form $\mathcal{D}^{J}_{m,a'}(g)$, where the state $|J, a', -n\rangle$ transforms as the J' representation of $SU(k)$. This can be constructed in terms of a product of the Abelian background and another representation of $SU(k+1)$. The state $|J, a', -n\rangle$ can be viewed as one set of states obtained by the reduction of the product $|J_1, -n\rangle \otimes |J_2, a', 0\rangle$ for some representations J_1, J_2 of $SU(k+1)$. In this way, matrices acting on the product space of two $SU(k+1)$ representations can lead to the symbols we obtained using the non-Abelian wavefunctions. This structure with two $SU(k+1)$ representations is what we expect for matter fields on $\mathbf{CP}^k$ which form an $SU(k+1)$ multiplet J_2, where one set (J_1 in our notation) arises from the translations on the space. This is also the mathematical structure relevant for the dynamics of a charged particle on a fuzzy space. (For example, for the fuzzy sphere, we find that two $SU(2)$ representations are needed to define charged particle dynamics with a constant (monopole) background field [33].) In summary, we see that dynamics in the LLL for smooth $\mathbf{CP}^k$ can reproduce dynamics on fuzzy $\mathbf{CP}^k$.

There is also a description of fuzzy $\mathbf{CP}^k$ directly in terms of embedding in $\mathbf{R}^{k^2+2k}$, which will be useful in our discussion. For this, we start with k^2+2k Hermitian matrices X_A which are of dimension $(N \times N)$, where N is of the form $(n+k)!/n!k!$ for some integer n. The embedding conditions are then given by [29, 34]

$$\begin{aligned} X_A X_A &= \frac{nk(n+k+1)}{2(k+1)} \equiv C_n, \\ d_{ABC} X_B X_C &= (k-1)\frac{(2n+k+1)}{4(k+1)} X_A \equiv \alpha_n X_A. \end{aligned} \tag{61}$$

Consider the $SU(k+1)$ generators T_A in the symmetric representation of rank n. They may be written as $T_A = a_a^\dagger (t_A)_{ab} a_b \equiv a^\dagger t_A a$, for bosonic annihilation–creation operators $a_b, a_a^\dagger, a, b = 1, \ldots, k+1$. By using completeness relations, one can easily prove that these obey representation-dependent identities which are identical to (61) with T_A replacing X_A. In other words, the matrices T_A in the symmetric rank n representation of $SU(k+1)$ give a solution of the embedding conditions (61) via $X_A = T_A = a^\dagger t_A a$. This solution is obviously fuzzy $\mathbf{CP}^k$ since functions of Xs become general $(N \times N)$-matrices, acting on the symmetric rank n representation of $SU(k+1)$. In equation (61), C_n is the quadratic Casimir operator and α_n is another invariant related to the properties of the d_{ABC} symbol. We may also note that conditions (61) can also be rewritten in terms of $-\mathrm{i}T_A$ as

$$\begin{aligned} (-\mathrm{i}T_A)(-\mathrm{i}T_A) &= -C_n, \\ d_{ABC}(-\mathrm{i}T_B)(-\mathrm{i}T_C) &= -\mathrm{i}\alpha_n(-\mathrm{i}T_A). \end{aligned} \tag{62}$$

A general gauge field is introduced in the matrix language by the prescription $D_A = -\mathrm{i}T_A + A_A$. This will involve k^2+2k spatial components for the gauge potential, which are obviously too many for $\mathbf{CP}^k$. Thus, there are restrictions on D_A which ensure that there are only $2k$ spatial components for the potentials. These conditions may be taken as the gauged version of conditions (62),

$$D_A D_A = -C_n, \qquad d_{ABC} D_B D_C = -\mathrm{i}\alpha_n D_A. \tag{63}$$

In other words, even after gauging, the derivatives obey the same embedding conditions (62) as before gauging [33]. (In the limit of a continuous manifold, there is some redundancy in these conditions. While they are sufficient for our purpose, whether they are necessary and sufficient in the noncommutative case is not quite a settled issue.)

9. The Chern–Simons action again

We will now reconsider the simplification of the action from a purely matrix point of view [35]. To carry out the expansion for large N, we write A_0, A_A in terms of $(N \times N)$-blocks. In other words, we can take $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_2$ so that the matrix elements of $A_A, i = 0, 1, 2, \dots,$ may be written as $A_{Apq} = \langle p|A_A|q\rangle = \langle la|A_A|rb\rangle, l, r = 1, 2, \dots, N, a, b = 1, 2, \dots, \dim J_2$. $\mathcal{H}_N$ will carry an irreducible representation of $SU(k+1)$, specifically the symmetric rank n representation. $\mathcal{H}_2$ carries the representation J_2 of some compact Lie group. (The action obtained in the previous sections is for the case when this group is $SU(k+1)$ or a subgroup of it.)

We will consider the variation of the matrix action $\mathrm{i}\,\mathrm{Tr}\, D_0$ under a change of the background fields. More generally, let K be a matrix acting on the Hilbert space $\mathcal{H}$. We may write K in an expansion in Ds as a sum of terms of the form

$$K = K(-\mathrm{i}T) = K^{A_1 A_2 \cdots A_s}(-\mathrm{i}T_{A_1})(-\mathrm{i}T_{A_2}) \cdots (-\mathrm{i}T_{A_s}). \tag{64}$$

(Our results extend by linearity to sums of such terms, so it is sufficient to consider one such term, for a fixed value of s.) In the large n limit, Ts typically become the coordinates for the space $\mathcal{M}$, in an embedding of $\mathcal{M}$ in $\mathbf{R}^d$ of suitable dimension d. Since $-\mathrm{i}T_A = D_A - A_A$, it is possible to expand K in terms of D_As which give the same basis on a background with additional gauge fields A_A. This can be done by writing $K = K(D_A - A_A)$ and expanding in powers of A. Since A is not necessarily small, it is easier to consider a perturbation around D and calculate the variation of K by expanding $K(D - \delta D) - K(D)$ to linear order in δD. By integrating this over δD_A up to A_A, we can obtain K. This will give an expression for K in terms of $K(D)$.

The actual calculation will involve a number of steps.

(i) We write the commutator of Ds as $[D_A, D_B] = f_{ABC} D_C + F_{AB} \equiv \omega_{AB} + F_{AB} \equiv \Omega_{AB}$. This defines ω, Ω. When the gauge field fluctuations are zero, $\omega_{AB} = f_{ABC} D_C$ will become the symplectic form in the large n limit. We first define a matrix N_{AC} which will play the role of an inverse to Ω_{CB} when acting on functions of Ds which obey the embedding conditions (63) and which tends to the inverse of the symplectic form at large n.

(ii) We then write δD in terms of N_{AC}, which will generate a series which is naturally in powers of $1/n$.

(iii) The variation of K to first order can then be obtained in a suitable form. In particular, we take $K = D_0$ to get the variation of the action, $\mathrm{i}\,\mathrm{Tr}(\delta D_0)$.

(iv) The next step will be to use the symbols to simplify the action. There is a correction to the definition of the symbol which must also be included.

(v) The result can be compared to the variation of the Chern–Simons action to establish that the action does indeed become the Chern–Simons action.

We shall now go over these steps, indicating briefly the basic mathematical results involved.

An 'inverse' to Ω. N_{AB} is defined by the equations

$$N_{AC}\Omega_{CB} = \delta_{AB} + \mathbb{X}_{AB} + \mathrm{i}\mathbb{Y}_{AB}, \qquad \Omega_{BC}N_{CA} = \delta_{BA} + \mathbb{X}_{BA} + \mathrm{i}\mathbb{Y}_{BA}, \tag{65}$$

where

$$\mathbb{X}_{AB} = \frac{D_B D_A}{B_n}, \qquad \mathbb{Y}_{AB} = \frac{1}{B_n}\left(n + \frac{1}{2}(k+1)\right)\left[d_{ABC}D_C + \mathrm{i}\frac{\alpha_n}{2}\delta_{AB}\right] \tag{66}$$

and B_n denotes the combination $B_n = \frac{1}{4}n(n+k+1) + \frac{1}{16}(k^2-1)$. We also introduce the expression

$$N_{0AC} = \frac{1}{B_n}\left[f_{ACK}D_K + \frac{1}{4}(k-1)\delta_{AC}\right]. \tag{67}$$

It obeys the equation $(N_0\omega)_{AB} = \delta_{AB} + \mathbb{X}_{AB} + \mathrm{i}\mathbb{Y}_{AB} + \mathbb{R}_{AB}$ with

$$\begin{aligned} B_n\mathbb{R}_{AB} = {} & f_{ACK}D_K f_{CBL}D_L - D_B D_A - \frac{C_n}{k}\delta_{AB} \\ & - \frac{\mathrm{i}}{2}(2n+k+1)d_{ABC}D_C + \frac{1}{4}(k-1)f_{ABC}D_C. \end{aligned} \tag{68}$$

The matrices $\mathbb{X}$ and $\mathbb{Y}$ are of order 1 at large n; $\mathbb{R}$ is naively of the same order, but it is actually of lower order due to algebraic identities on T_A. A solution for N_{AB} can be obtained as a series by writing $N = N_0 + N_1 + N_2 + \cdots$ and matching terms of the same order in powers of n in (65). The first few terms are given by

$$\begin{aligned} N_{AB} = {} & N_{0AB} - (\mathbb{R}N_0)_{AB} - (N_0FN_0)_{AB} + (\mathbb{R}(\mathbb{X}+\mathrm{i}\mathbb{Y})N_0)_{AB} \\ & + (N_0F(\mathbb{X}+\mathrm{i}\mathbb{Y})N_0)_{AB} + (N_0FN_0FN_0)_{AB} + \cdots. \end{aligned} \tag{69}$$

The embedding conditions (63) are crucial in verifying that this is a solution to (65). Terms containing powers of $\mathbb{X}$, $\mathbb{Y}$, such as $\mathbb{R}\mathbb{X}N_0$, $\mathbb{R}\mathbb{Y}N_0$, are seemingly of the same order $\mathbb{R}N_0$, since $\mathbb{X}$, $\mathbb{Y}$ are of order 1. But they are actually down by a power of n due to the embedding conditions. The series (69), therefore, is appropriate at large n.

Expression for δD. Using N_{AB}, we can express the variation of D in a form suitable for a series expansion in $1/n$. Multiplying equations (65) by δD_A, using $\delta D_A\mathbb{Y}_{AB} + \mathbb{Y}_{BA}\delta D_A = 0$, which also follows from the embedding conditions, and rearranging terms one can show that

$$\begin{aligned} &\delta D_B = \frac{1}{2}(\xi_C[D_C, D_B] + [D_B, D_C]\tilde{\xi}_C) - \frac{1}{4B_n}[\delta D\cdot D - D\cdot\delta D, D_B], \\ &\xi_C = \delta D_A\left(N_{AC} + \frac{\delta_{AC}}{B_n}\right), \qquad \tilde{\xi}_C = \left(N_{CA} + \frac{\delta_{CA}}{B_n}\right)\delta D_A. \end{aligned} \tag{70}$$

Variation of K. We now turn to the matrix function $K = K^{A_1A_2\cdots A_s}D_{A_1}D_{A_2}\cdots D_{A_s}$. Here we can take, without loss of generality, the coefficients $K^{A_1A_2\cdots A_s}$ to be symmetric in all indices. (Any antisymmetric pair may be reduced to a single D and F; F itself may be re-expanded in terms of Ds, to bring it to this form.) Taking the variation of K under $D\to D-\delta D$ and rearranging terms keeping in mind this symmetry, we can bring δK to the form

$$\mathrm{Tr}(\delta K) = -\tfrac{1}{2}\,\mathrm{Tr}[\delta D_A N_{AB}[D_B, K] - [D_B, K]N_{BA}\delta D_A + \mathrm{O}(1/n^3)]. \tag{71}$$

This gives the change in $\mathrm{Tr}\,K$ to order $1/n^2$, as n becomes large.

Variation of $\mathrm{Tr}(D_0)$. Equation (71) can be used to work out the expansion of the action, at least to order $1/n^2$, taking $K = D_0$. The terms in N_{AB}, from equation (69), which can contribute to this order, are

$$N_{AB} = \omega^{-1}_{AB} + \frac{(k-1)}{4B_n}\delta_{AB} - \omega^{-1}_{AC}F_{CD}\omega^{-1}_{DB} - \mathbb{R}_{AC}\omega^{-1}_{CB} + \mathrm{O}(1/n^3), \tag{72}$$

where $\omega^{-1}_{AB} = f_{ABC} D_C / B_n$. (The notation anticipates the fact that this matrix will become the inverse of ω in the large n limit. But, at this stage ω^{-1}_{AB} is still a matrix.) The variation of the action, up to order $1/n^2$, is obtained from (71), (72) as

$$\begin{aligned} \mathrm{Tr}(\delta D_0) &= \delta D_0^{(1)} + \delta D_0^{(2)} + \cdots, \\ \delta D_0^{(1)} &= -\tfrac{1}{2}\big(\delta A_A \omega^{-1}_{AB} F_{B0} + F_{B0}\omega^{-1}_{AB}\delta A_A\big), \\ \delta D_0^{(2)} &= \tfrac{1}{2}\big(\delta A_A \omega^{-1}_{AB} F_{BC}\omega^{-1}_{CD} F_{D0} + F_{D0}\omega^{-1}_{AB} F_{BC}\omega^{-1}_{CD}\delta A_A\big). \end{aligned} \tag{73}$$

Using symbols. What is left is to simplify the expansion (73), which is still in matrix terms, in terms of the symbols as an integral over $\mathbf{CP}^k$ with a trace over the remaining (small, a, b type) matrix labels. For this, we can bring ω^{-1}_{AB} to the left end by the cyclicity of the trace, and then replace it by

$$\omega^{-1}_{AB} = \frac{1}{B_n}\left[-\mathrm{i}\frac{nk}{\sqrt{2k(k+1)}} f_{ABC} S_{C,k^2+2k} + \frac{\mathrm{i}}{2} f_{ABC} S_{C-i} R_{+i} + f_{ABC} S_{C\alpha} A_\alpha\right]. \tag{74}$$

Here we have used the standard rule for simplifying the symbol of $T_A X$ and also the fact the symbol of the gauge field may be written as $A_C = S_{C\alpha} A_\alpha$ where the summation is over $\alpha = 1$ to $2k$. The fact that the symbol of A_C has this restricted form is due to the constraints (63). The inverse of ω, in the limit of the continuous manifold, is given in the coordinate basis as

$$\omega^{-1ij} = -\mathrm{i}\frac{nk}{B_n\sqrt{2k(k+1)}} f^{\alpha\beta,k^2+2k} (E^{-1})^i_\alpha (E^{-1})^j_\beta, \tag{75}$$

where Es are the frame fields for the metric on $\mathbf{CP}^k$. This can be used to simplify the first term on the right-hand side of (74) as $\omega^{-1ij} E^\alpha_i E^\beta_j S_{A\alpha} S_{B\beta}$.

Change in symbol. There is one more correction which we must take account of. The symbol was defined using wavefunctions with the gauge field fluctuations equal to zero. As the potential is changed, the definition of the symbol also changes. This change can be calculated as

$$(K) = (K)_0 - \tfrac{1}{4}\big[(\omega^{-1}_{AB} F_{AB} + F_{AB}\omega^{-1}_{AB})K\big] + \cdots. \tag{76}$$

This is needed to simplify the symbol for $\delta D_0^{(1)}$, for which $K = -\frac{1}{2}\big(\delta A_A \omega^{-1}_{AB} F_{B0} + F_{B0}\omega^{-1}_{AB}\, \delta A_A\big)$.

The variation of the action. Taking account of these observations, the evaluation of the action is straightforward, although somewhat tedious. The result, to order $1/n^2$, is obtained as

$$\begin{aligned} \int \mathrm{d}t\, \mathrm{Tr}(\delta D_0) = N \int \mathrm{d}t\, \mathrm{d}\mu(\mathbf{CP}^k)\Bigg[&-\omega^{-1ij}\,\mathrm{tr}(\delta A_i F_{j0}) \\ &+ \frac{1}{2}\left(\omega^{-1im}\omega^{-1nj} + \frac{1}{2}\omega^{-1ij}\omega^{-1mn}\right)\mathrm{tr}[(\delta A_i F_{j0} + F_{j0}\delta A_i)F_{mn}] \\ &- \frac{1}{n}\omega^{-1mn}\,\mathrm{tr}[\delta A_m(-D^2 + (k+1))F_{n0}] + \mathrm{O}(1/n^3)\Bigg]. \end{aligned} \tag{77}$$

In this expression, the field components are in the coordinate basis.

Relation to the Chern–Simons action. Equation (77) can be expressed in terms of the Chern–Simons form. The variation of the $(2k+1)$-dimensional Chern–Simons term is given by

$$\delta S = \frac{\mathrm{i}^{k+1}}{(2\pi)^k k!}\int \mathrm{tr}(\delta A F^k). \tag{78}$$

Replacing F by $\omega + F$ and expanding, we get

$$\delta S = \frac{\mathrm{i}^{k+1}}{(2\pi)^k k!} \int \mathrm{tr}\left(\omega^k \delta A + k\omega^{k-1}\delta A F + \frac{1}{2}k(k-1)\omega^{k-2}\delta A F^2 + \cdots\right). \tag{79}$$

Since ω is proportional to the Kähler form, this can be simplified and written in terms of the standard volume measure $\mathrm{d}\mu$ for $\mathbf{CP}^k$. As an example, we note that the second term can be written as

$$\frac{\mathrm{i}^{k+1}}{(2\pi)^k k!} k\omega^{k-1}\, \mathrm{tr}(\delta A_m F_{n0})\, \mathrm{d}x^m\, \mathrm{d}x^n\, \mathrm{d}t = \mathrm{i}\frac{n^k}{k!} \int \mathrm{d}t\, \mathrm{d}\mu[-\omega^{-1mn}\ \mathrm{tr}(\delta A_m F_{n0})]. \tag{80}$$

Rewriting the other terms similarly, and comparing with (77), we find

$$\begin{aligned} \mathrm{i} \int \mathrm{d}t\, \mathrm{Tr}(\delta D_0) &= \frac{Nk!}{n^k}\delta S_{\mathrm{CS}} - \mathrm{i}\frac{N}{n} \int \mathrm{d}t\, \mathrm{d}\mu\, \omega^{-1mn}\ \mathrm{tr}[\delta A_m(-D^2 + (k+1))F_{n0}] + \cdots \\ &\approx \delta S_{\mathrm{CS}} - \mathrm{i}\frac{N}{n} \int \mathrm{d}t\, \mathrm{d}\mu\, \omega^{-1mn}\ \mathrm{tr}[\delta A_m(-D^2 + (k+1))F_{n0}] + \mathrm{O}(1/n^3). \end{aligned} \tag{81}$$

In this expression for $\mathrm{i}\,\mathrm{Tr}(D_0)$, we have also included a term

$$\mathrm{i}\,\mathrm{Tr}\, \delta_f A_0 = \frac{Nk!}{n^k} \frac{\mathrm{i}^{k+1}}{(2\pi)^k k!} \int \omega^k\, \mathrm{tr}(\delta A_0). \tag{82}$$

The reason is that (77) only gives the variation of $\mathrm{i}\,\mathrm{Tr}(D_0)$ due to the change in the spatial components of A, namely under $A_i \to A_i + \delta A_i$. A change in the functional form of A_0 is also possible; the variation of $\mathrm{i}\,\mathrm{Tr}(D_0)$ due to this is (82) and should be included to obtain the general variation.

The first term on the right-hand side of (81) will integrate to give the Chern–Simons form. The second term is due to the higher terms, terms involving derivatives, in the star product. Let $S_{*\mathrm{CS}}$ denote the Chern–Simons term defined with star products used for the products of fields and their derivatives occurring in it. The integrated version of (81) is then

$$\mathrm{i} \int \mathrm{d}t\, \mathrm{Tr}(D_0) \approx S_{*\mathrm{CS}} + \cdots. \tag{83}$$

It is worth emphasizing that the gauge potentials in this Chern–Simons action are the full potentials $a + A$, where a is the background value corresponding to the symplectic form and A is the additional potential or gauge field fluctuation.

The second term in (81) which arises from higher terms in the star product also agrees with the result (55), if we write $A_0 = \mathrm{i}V$; some partial integrations are also necessary. As argued after (55), by rescaling the coordinates $x \to \tilde{x} = Rx$, this term is seen to be small if the radius is large and the gradients of fields are small compared to the value of the background field. The Chern–Simons form is unaffected by this scaling. In this approximation, the leading term of the action is given by

$$\mathrm{i} \int \mathrm{d}t\, \mathrm{Tr}(D_0) \approx S_{\mathrm{CS}} + \cdots. \tag{84}$$

This result shows that the expansion of $\mathrm{Tr}(D_0)$ around different backgrounds can be approximated, in the large n limit and for small gradients for the field strengths, by the Chern–Simons form, with A replaced by $a + A$, a being the desired background potential. Note that any reference to the metric and other geometrical properties of $\mathbf{CP}^k$ has disappeared in this result.

Strictly speaking, our calculation has explicitly verified this result (84) only up to order $1/n^2$ or, equivalently, up to the term involving the five-dimensional Chern–Simons form for A. To this order, we do get $S_{\mathrm{CS}}(a + A)$. But we can see that the result (84) holds with the full Chern–Simons term. This is because, the final expression, whatever it is, should be a functional of only the combination $a + A$, since the separation between the background and fluctuation is arbitrary. Also it should have the correct gauge invariance property and it should agree with $S_{\text{CS}}(a+A)$ when expanded up to the term with the five-dimensional Chern–Simons form for A. The only such term, apart from the ambiguity of higher gradients of fields, is $S_{\text{CS}}(a + A)$. Thus, the result (84) holds, in general, in a gradient expansion at large R.

10. Towards a matrix theory of gravity

The basic mathematical result we have can be applied to gravity on fuzzy spaces [36]. As mentioned in the introduction, one may take the minimalist point of view that fuzzy spaces are just another regularization. Then, for ordinary field theories it would not be anything special, but it is still very attractive for gravity since symmetries can be preserved. One may also consider fuzzy space as fundamental, continuous space being a large N approximation. Formulating gravity on fuzzy spaces is, in either case, an interesting problem. (For earlier formulations of gravity on fuzzy spaces, see [6, 37].)

To see how gravity arises naturally, recall that the background fields we have considered are valued in the Lie algebra of $U(k)$ for $\mathbf{CP}^k = SU(k+1)/U(k)$. This is part of the isometry group $SU(k + 1)$ of $\mathbf{CP}^k$. Gauging of the isometry group introduces gravity, so we may interpret the gauge fields as gravitational fields. This is the basic point of contact. We shall now present an argument on how our results may be adapted for describing gravity.

The setting for the problem is the finite-dimensional Hilbert space $\mathcal{H}$, which we may take to be split into a matter part $\mathcal{H}_m$ and a space part $\mathcal{H}_s$, the latter leading to the spacetime at large n. The action for the evolution of states is given by the action

$$S = \mathrm{i}\int \mathrm{d}t\, \mathrm{Tr}\rho_0(U^\dagger D_0 U), \tag{85}$$

where $D_0 = \partial_0 + A_0$. The Hamiltonian as a matrix on the Hilbert space is $H = -\mathrm{i}A_0$. We know that (85) is the most general equation for evolution of states for matter, the specific characteristics of the matter system being encoded in the choice of H and other observables. The only natural choice for the space part is that the same action should apply to evolution within $\mathcal{H}_s$. To see how this can be implemented, represent a general state in $\mathcal{H}$ as $|A, r\rangle$, where the labels A, B, etc pertain to the degrees of freedom of space and the labels r, s, etc describe the matter system of interest. For the operator D_0, we introduce the splitting

$$\langle A, r|D_0|B, s\rangle = \delta_{rs}\langle A|D_0^{(s)}|B\rangle + \langle A, r|D_0^{(m)}|B, s\rangle. \tag{86}$$

The part of D_0 which is proportional to the identity in $\mathcal{H}_m$ is designated as $D_0^{(s)}$ and the remainder as $D_0^{(m)}$. The latter includes effects of coupling the matter system of interest to the spatial degrees of freedom. We also take the density matrix to have the form

$$\langle A, r|\rho_0|B, s\rangle = \delta_{AB}\langle r|\rho_0|s\rangle. \tag{87}$$

Note that we take ρ_0 to have maximal rank in $\mathcal{H}_s$; if the rank is less than maximal, it would mean that the dynamics does not cover all of space. This is why we make the choice (87).

While A_0 (or H) specifies the choice of matter system, for spacetime, the geometry is not *a priori* determined, therefore $D^{(s)}$ should be regarded as an arbitrary matrix in (85). Thus, we take the action (85) as the action for the theory, including gravity, where U and $D_0^{(s)}$ are regarded as quantities to be varied. $D_0^{(m)}$ is to be regarded as a given operator, specifying the matter system of interest.

We can get different large n limits for the action depending on what backgrounds we choose. Extremization of the action can be used to determine the best background. If we ignore all matter degrees of freedom as a first approximation, the action becomes

$$S \approx \mathrm{i} \int \mathrm{d}t\, \mathrm{Tr}\big(D_0^{(s)}\big). \tag{88}$$

For the case of $\mathbf{CP}^k$ with a non-Abelian background, the wavefunctions were of the form $\mathcal{D}^J_{m,a'}(g)$. As mentioned in section 8, the state $|J, a', -n\rangle$ may be taken as one set of states obtained by the reduction of the product $|J_1, -n\rangle \otimes |J_2, a', 0\rangle$ for some representations J_1, J_2 of $SU(k+1)$. (We will take J_2 to be the fundamental representation of $SU(k+1)$ for simplicity.) Therefore, at the matrix level we split the states as $\mathcal{H}_s = \mathcal{H}_N \otimes \mathcal{H}_2$, where the components are of dimensions $N = \dim J_1$ and $\dim J_2 = k+1$, respectively. Correspondingly, we write D_0 as $D_{0pq} = \langle p|D_0|q\rangle = \langle la|D_0|rb\rangle, l, r = 1, 2, \ldots, N, a, b = 1, 2, \ldots, k+1$. The matrix structure for the indices l, r will be converted to the symbol, and we carry out a large N expansion. The result is $(2k+1)$-dimensional Chern–Simons theory. For the gauge fields, in general, there will be a $U(1)$ component as well. Thus, the gauge group is $U(k+1)$. The conclusion of this argument is that, for a fuzzy space, we should expect Chern–Simons gravity [38]. (In this context, it is fascinating that there are indications of Chern–Simons gravity in the context of M-theory [39]; we expect that the present analysis can be related to a matrix version of some of the considerations in these references.)

The simplest example along these lines would be $k = 3$, which gives a $U(4)$ Chern–Simons theory on a seven-dimensional space. We take this space to be of the form $S^2 \times M^5$ and the gauge field strength as $-\mathrm{i}l\omega_K + F$, where ω_K is the Kähler form on S^2, l is an integer and F is in the $SU(4)$ Lie algebra. The action is then given by

$$S = -\mathrm{i}\frac{l}{24\pi^2} \int \mathrm{tr}\left(A\, \mathrm{d}A\, \mathrm{d}A + \frac{3}{2}A^3\, \mathrm{d}A + \frac{3}{5}A^5\right). \tag{89}$$

Since $SU(4)$ is locally isomorphic to $O(6)$, we see that this is appropriate for Euclidean gravity in five dimensions. In fact, the $SU(4)$ potential can be written as $A = P^\alpha E^\alpha_i\, \mathrm{d}x^i + \frac{1}{2}J^{\alpha\beta}\omega^{\alpha\beta}_i\, \mathrm{d}x^i$ where $J^{\alpha\beta}$ are the generators of $O(5) \subset O(6)$ and P^α are a basis for the complement of $\underline{O(5)}$ in $\underline{O(6)}$. E^α are the frame fields and $\omega^{\alpha\beta}$ is the spin connection. The equations of motion give $F = 0$ and have the solution $A = g^{-1}\,\mathrm{d}g$ with $g \in O(6)$. This is with no matter. This space is $O(6)/O(5) = S^5$ which is the Euclidean version of de Sitter space. It is given in a basis where the cosmological constant Λ has been scaled out; it may be introduced by the replacement $E^\alpha \to \sqrt{\Lambda}E^\alpha$.

There is also a neat reduction of this to four dimensions [40]. This is achieved by the additional restrictions $E^5_5 = 1, \omega^{5\beta} = 0, \omega^{\alpha\beta}_5 = 0$, for $\alpha, \beta = 1, \ldots, 4$. The fifth dimension is taken as a circle of, say, unit radius. (This restriction, as well as the choice of S^2 with the $U(1)$ field proportional to the Kähler form, can be interpreted as particular compactifications.) In this case, the action becomes

$$\begin{aligned} S &= \frac{l\Lambda}{64\pi} \int \left(E^\alpha E^\beta R^{\gamma\delta} - \frac{\Lambda}{2}E^\alpha E^\beta E^\gamma E^\delta\right)\epsilon_{\alpha\beta\gamma\delta} \\ &= \frac{l\Lambda}{16\pi} \int \mathrm{d}^4x \sqrt{g}(R - 3\Lambda). \end{aligned} \tag{90}$$

We get the Einstein–Hilbert action with a cosmological constant.

There are many issues, such as getting Minkowski signature, generalizing to other dimensions and incorporating matter in a detailed way, which are not yet clear. Nevertheless, this nexus of Hall effect, fuzzy spaces and gravity is very suggestive and intriguing.

11. Discussion and outlook

We shall begin with a synopsis of the basic result of our analysis. We have a finite-dimensional Hilbert space $\mathcal{H}$ with fields and other observables realized as linear transformations or matrices acting on this space. When the dimension of the matrices is large, we can simplify matrix products, the action, etc, using symbols for the matrices and star products. The symbols are defined in terms of a set of wavefunctions. These wavefunctions are based on a continuous smooth symplectic space $\mathcal{M}$, of dimension, say, $2k$, with a set of gauge fields defined on it. (We may think of the Hilbert space $\mathcal{H}$ as providing a fuzzy version of $\mathcal{M}$.) The wavefunctions characterize the space $\mathcal{M}$ and the fields on it, as far as observables are concerned. The large N limits of matrices are thus parametrized by $\mathcal{M}$ and the gauge fields. Alternatively, we may think of the Hilbert space as the set of lowest Landau levels for quantum Hall effect on $\mathcal{M}$ and the gauge fields as external fields to which the fermions couple. In either case, the basic action we have analysed is of the form

$$S = \mathrm{i} \int \mathrm{d}t \, \mathrm{Tr}(\hat{\rho}_0 \hat{U}^\dagger \hat{D}_0 \hat{U}), \tag{91}$$

where $\hat{\rho}_0$ characterizes a fiducial or initial state. Our basic result is then the following.

- In the large N limit, $\hat{\rho}_0$ describes a droplet of a subspace of $\mathcal{M}$. The simplification of the action yields a bulk action and a boundary action.
- The bulk action is given by the $(2k + 1)$-dimensional Chern–Simons action, when the gradients of the gauge fields are small. This action, although obtained by expansion around a chosen background, is not sensitive to the geometry of the space.
- The boundary action describes the fluctuations of the boundary of the droplet or, equivalently, the large N limit of embeddings of a fuzzy sphere in the fuzzy version of $\mathcal{M}$. It is given by a chiral, gauged higher dimensional generalization of the WZW action.
- The bulk and boundary actions are not separately gauge invariant, but the total action is, with the gauge anomalies cancelling between the two.

Perhaps, the most interesting conclusion which emerges from this analysis is the possibility of describing a number of higher dimensional theories as matrix models. For example, the $(2 + 1)$-dimensional Chern–Simons and the two-dimensional WZW theories help to define conformal field theories in two dimensions. One can introduce a matrix model for them, as a specific large N limit of (91). But such a matrix model can also lead to higher dimensional Chern–Simons and WZW models, as a different way of taking the large N limit. This suggests a way of generalizing conformal field theories to higher dimensions. In this context, the exploration of some of the well-known features of WZW models such as symmetry structures and current algebra would be very interesting. (This is also closely related to the Kähler–Chern–Simons and Kähler–WZW models [31].)

Noncommutative Chern–Simons theories have been extensively investigated over the last few years [41]. Properties of such theories on flat noncommutative spaces are fairly well understood by now. They have also been formulated on some $(2 + 1)$-dimensional fuzzy spaces, but a general formulation on fuzzy spaces has not yet been possible [42]. These theories are matrix versions of the Chern–Simons theory characterized by the choice of the fuzzy space and the gauge group and give the usual Chern–Simons theory at large N, just as the action (91) does. It is perfectly sensible to study these matrix models and (91) as different theories, but if we only ask for a matrix theory whose commutative limit gives the Chern–Simons theory, then the action (91) is a good choice. It has also the advantage that it can easily be formulated on any fuzzy space and can give Chern–Simons theories on smooth spaces of any dimension and with any gauge group, depending on how the large N limit is taken. The matrix version (91) may thus be considered as a 'universal' Chern–Simons theory [35].

Bosonization in higher dimensions is another closely related topic [43, 44]. The phase space for fermions in k dimensions is $2k$-dimensional. Semiclassically, each quantum state corresponds to a certain phase volume. Thus, by the exclusion principle, a collection of a large number of fermions is an incompressible droplet in the phase space. Deformations of the droplet give a bosonic description of the dynamics of this collection of fermions. Thus, the matrix action (91) can also be used for phase space bosonization. The large N result is evidently the generalized WZW theory. Related approaches in formulating higher dimensional phase space bosonization in terms of a noncommutative field theory have been explored in [20, 44].

There is also an evident connection to fluid dynamics; the edge dynamics of the droplet is that of an incompressible droplet of fluid. The additional gauge fields allow for nonzero compressibility. The droplet can be viewed as the embedding of a $2k$-brane in $\mathcal{M}$. Therefore, one should be able to relate these ideas to the descriptions of fluids in the brane language [45]. It is also related to the noncommutative description of the quantum Hall effect proposed in [46].

As mentioned in the previous section, gravity on a fuzzy space may be the context in which these results can be most fruitful. This story is far from complete, there are many issues related to the Minkowski signature, incorporation of matter, etc which need to be clarified. Also, as mentioned earlier, there are suggestions that Chern–Simons gravity can provide an effective description of M-theory [39]. A matrix description via (91) is an attractive possibility that needs to be explored further. It is also suggestive that quantum Hall droplets appear in the dual field theories for many gravitational backgrounds [47].

Another interesting line of development, which we have not discussed, is the supersymmetric version of quantum Hall effect [48]. The bosonic partners of fermions do not have to form an incompressible droplet since there is no exclusion principle for them. Nevertheless, it is possible to obtain supersymmetric droplets and study their properties. This can be applicable in the context of supersymmetric brane dynamics, supergravity, etc.

Acknowledgments

This work was supported in part by the National Science Foundation grants PHY-0457304 and PHY-0244873 and by PSC-CUNY grants.

References

[1] For a recent review of fuzzy spaces and theories defined on them, see Balachandran A P 2002 *Pramana* **59** 359
Balachandran A P and Kurkcuoglu S 2004 *Int. J. Mod. Phys.* A **19** 3395
Balachandran A P, Kurkcuoglu S and Vaidya S 2005 *Lectures on Fuzzy and Fuzzy SUSY Physics Preprint* hep-th/0511114

[2] Balachandran A P, Govindarajan T R and Ydri B 2000 *Mod. Phys. Lett.* A **15** 1279
Balachandran A P, Govindarajan T R and Ydri B 2000 *Preprint* hep-th/0006216
Aoki H, Iso S and Nagao K 2003 *Phys. Rev.* D **67** 065018

[3] Connes A 1994 *Noncommutative Geometry* (New York: Academic)
Madore J 2000 *An Introduction to Noncommutative Differential Geometry and Physical Applications (LMS Lecture Notes Series 257)* (Cambridge: Cambridge University Press)
Landi G 1997 *An Introduction to Noncommutative Spaces and Their Geometry (Lecture Notes in Physics, Monographs m51)* (Berlin: Springer)

[4] For a review of the matrix version of M-theory and its solutions, see Taylor W IV 2001 *Rev. Mod. Phys.* **73** 419

[5] For a review of field theories on noncommutative spaces, see Douglas M R and Nekrasov N A 2001 *Rev. Mod. Phys.* **73** 977

[6] Chamseddine A, Felder G and Fröhlich J 1993 *Commun. Math. Phys.* **155** 205
Kalau W and Walze M 1995 *J. Geom. Phys.* **16** 327
Kastler D 1995 *Commun. Math. Phys.* **166** 633
Chamseddine A and Connes A 1997 *Commun. Math. Phys.* **186** 731
Madore J 1997 *Preprint* gr-qc/9709002
Ho P-M 1997 *Int. J. Mod. Phys.* A **12** 923
Moffat J 2000 *Phys. Lett.* B **491** 345
Banados M *et al* 2001 *Phys. Rev.* D **64** 084012
Cacciatori S *et al* 2002 *Class. Quantum Grav.* **19** 4029
Viet N A and Wali K C 2003 *Phys. Rev.* D **67** 124029
Chamseddine A 2003 *J. Math. Phys.* **44** 2534
Castro C 2004 *Gen. Rel. Grav.* **36** 2605
Aschieri P *et al* 2005 *Class. Quantum Grav.* **22** 3511
Aschieri P *et al* 2006 *Class. Quantum Grav.* **23** 1883
Balachandran A P, Govindarajan T R, Gupta K S and Kurkcuoglu S 2006 *Preprint* hep-th/0602265

[7] de Wit B, Hoppe J and Nicolai H 1988 *Nucl. Phys.* B **305** 545

[8] Banks T, Fischler W, Shenker S and Susskind S 1997 *Phys. Rev.* D **55** 5112
Ishibashi N, Kawai H, Kitazawa Y and Tsuchiya A 1997 *Nucl. Phys.* B **498** 467

[9] There are many papers on this; the paper Berenstein D, Maldacena J M and Nastase H 2002 *J. High Energy Phys.* JHEP04(2002)013 and citations to it can give a perspective on matrices and branes

[10] Karabali D, Nair V P and Randjbar-Daemi S 2004 *From Fields to Strings: Circumnavigating Theoretical Physics, Ian Kogan Memorial Collection* ed M Shifman, A Vainshtein and J Wheater (Singapore: World Scientific) pp 831–76

[11] For a recent review on QHE, see Girvin S 2000 *Topological Aspects of Low Dimensional Systems* ed A Comtet *et al* (Berlin: Springer) (*Preprint* cond-mat/9906454) and references therein

[12] Zhang S C and Hu J P 2001 *Science* **294** 823
Hu J P and Zhang S C 2001 *Preprint* cond-mat/0112432

[13] Karabali D and Nair V P 2002 *Nucl. Phys.* B **641** 533

[14] Karabali D and Nair V P 2004 *Nucl. Phys.* B **679** 427
Karabali D and Nair V P 2004 *Nucl. Phys.* B **697** 513

[15] Fabinger M 2002 *J. High Energy Phys.* JHEP05(2002)037
Chen Y X, Hou B Y and Hou B Y 2002 *Nucl. Phys.* B **638** 220
Kimura Y 2002 *Nucl. Phys.* B **637** 177
Bernevig B A, Chern C H, Hu J P, Toumbas N and Zhang S C 2002 *Ann. Phys., NY* **300** 185
Zhang S C 2003 *Phys. Rev. Lett.* **90** 196801
Dolan B 2003 *J. High Energy Phys.* JHEP05(2003)18
Meng G 2003 *Int. J. Mod. Phys.* A **36** 9415
Bellucci S, Casteill P Y and Nersessian A 2003 *Phys. Lett.* B **574** 121
Hasebe K and Kimura Y 2004 *Phys. Lett.* B **602** 255

[16] Elvang H and Polchinski J 2002 *Preprint* hep-th/0209104

[17] Nair V P and Randjbar-Daemi S 2004 *Nucl. Phys.* B **679** 447

[18] Bernevig B A, Hu J P, Toumbas N and Zhang S C 2003 *Phys. Rev. Lett.* **91** 236803

[19] Jellal A 2005 *Nucl. Phys.* B **725** 554
Daoud M and Jellal A 2006 *Preprint* hep-th/0605289

[20] Polychronakos A P 2005 *Nucl. Phys.* B **705** 457
Polychronakos A P 2005 *Nucl. Phys.* B **711** 505

[21] Haldane F D M 1983 *Phys. Rev. Lett.* **51** 605

[22] Karabali D 2005 *Nucl. Phys.* B **726** 407
Karabali D 2006 *Preprint* hep-th/0605006

[23] Iso S, Karabali D and Sakita B 1992 *Phys. Lett.* B **296** 143
Cappelli A, Dunne G, Trugenberger C and Zemba G 1993 *Nucl. Phys.* B **398** 531
Cappelli A, Trugenberger C and Zemba G 1993 *Nucl. Phys.* B **396** 465
Cappelli A, Trugenberger C and Zemba G 1994 *Phys. Rev. Lett.* **72** 1902
Karabali D 1994 *Nucl. Phys.* B **419** 437
Karabali D 1994 *Nucl. Phys.* B **428** 531
Flohr M and Varnhagen R 1994 *J. Phys. A: Math. Gen.* **27** 3999

[24] Das S R, Dhar A, Mandal G and Wadia S R 1992 *Int. J. Mod. Phys.* A **7** 5165
Das S R, Dhar A, Mandal G and Wadia S R 1992 *Mod. Phys. Lett.* A **7** 71
Dhar A, Mandal G and Wadia S R 1993 *Int. J. Mod. Phys.* A **8** 325
Dhar A, Mandal G and Wadia S R 1992 *Mod. Phys. Lett.* A **7** 3129
Dhar A, Mandal G and Wadia S R 1993 *Mod. Phys. Lett.* A **8** 3557
Dhar A 2005 *J. High Energy Phys.* JHEP07(2005)064
[25] Sakita S 1993 *Phys. Lett.* B **315** 124
Sakita B 1996 *Phys. Lett.* B **387** 118
Sakita B and Ray R 2001 *Phys. Rev.* B **65** 035320
[26] A similar approach, also for two-dimensional QHE, but based on W_∞-transformations mixing higher Landau levels has been proposed by Shizuya K 1995 *Phys. Rev.* B **52** 2747
[27] Seiberg N and Witten E 1999 *J. High Energy Phys.* JHEP09(1999)032
[28] Madore J, Schraml S, Schupp P and Wess J 2000 *Eur. Phys. J.* C **16** 161
Jurco B, Moller L, Schraml S, Schupp P and Wess J 2001 *Eur. Phys. J.* C **21** 383
Behr W and Sykora A 2004 *Nucl. Phys.* B **698** 473
[29] Alexanian G, Balachandran A P, Immirzi G and Ydri B 2002 *J. Geom. Phys.* **42** 28
Balachandran A P, Dolan B P, Lee J, Martin X and O'Connor D 2002 *J. Geom. Phys.* **43** 184
[30] Brown L S and Nepomechie R I 1987 *Phys. Rev.* D **35** 3239
Karabali D, Park Q H, Schnitzer H J and Yang Z 1989 *Phys. Lett.* B **216** 307
Karabali D and Schnitzer H J 1990 *Nucl. Phys.* B **329** 649
[31] Nair V P and Schiff J 1990 *Phys. Lett.* B **246** 423
Nair V P and Schiff J 1992 *Nucl. Phys.* B **371** 329
See also Donaldson S 1985 *Proc. London Math. Soc.* **50** 1
Losev A *et al* 1996 *Nucl. Phys.* B **46** 130
Inami T, Kanno H and Ueno T 1997 *Mod. Phys. Lett.* A **12** 2757
[32] Wen X G 1990 *Phys. Rev.* B **41** 12838
Lee D H and Wen X G 1991 *Phys. Rev. Lett.* **66** 1765
Stone M 1990 *Phys. Rev.* B **42** 8399
Stone M 1991 *Ann. Phys., NY* **207** 38
Frohlich J and Kepler T 1991 *Nucl. Phys.* B **354** 369
[33] Karabali D, Nair V P and Polychronakos A P 2002 *Nucl. Phys.* B **627** 565
Grosse H and Steinacker H 2005 *Nucl. Phys.* B **707** 145
[34] Nair V P and Randjbar-Daemi S 1998 *Nucl. Phys.* B **533** 333
Grosse H and Strohmaier A 1999 *Lett. Math. Phys.* **48** 163
Alexanian G, Pinzul A and Stern A 2001 *Nucl. Phys.* B **600** 531
[35] Nair V P 2006 *Preprint* hep-th/0605007
[36] Nair V P 2006 *Preprint* hep-th/0605008
[37] Nair V P 2003 *Nucl. Phys.* B **651** 313
Abe Y and Nair V P 2003 *Phys. Rev.* D **68** 025002
[38] Chamseddine A H 1989 *Phys. Lett.* B **233** 291
Chamseddine A H 1990 *Nucl. Phys.* B **346** 213
Witten E 1988 *Nucl. Phys.* B **311** 46
Achucarro A and Townsend P 1986 *Phys. Lett.* B **180** 89
For a recent review on Chern–Simons gravity, see Zanelli J 2000 *Braz. J. Phys.* **30** 251
Zanelli J 2005 *Preprint* hep-th/0502193
[39] Troncoso R and Zanelli J 1998 *Phys. Rev.* D **58** 101703
Horava P 1999 *Phys. Rev.* D **59** 046004
Nastase H 2003 *Preprint* hep-th/0306269
Horava P and Keeler C A 2005 *Preprint* hep-th/0508024
[40] This is related to the description of gravity developed in Mansouri F and Chang L N 1976 *Phys. Rev.* D **13** 3192
MacDowell S and Mansouri F 1977 *Phys. Rev. Lett.* **38** 739
Mansouri F 1977 *Phys. Rev.* D **16** 2456
[41] Chamseddine A H and Frohlich J 1994 *J. Math. Phys.* **35** 5195
Krajewski T 1998 *Preprint* math-ph/9810015
Mukhi S and Suryanarayana N V 2000 *J. High Energy Phys.* JHEP11(2000)006
Polychronakos A P 2000 *J. High Energy Phys.* JHEP11(2000)008
Chen G H and Wu Y S 2001 *Nucl. Phys.* B **93** 562
Grandi N V and Silva G A 2001 *Phys. Lett.* B **507** 345

[42] Morariu B and Polychronakos A P 2005 *Phys. Rev.* D **72** 125002
[43] Luther A 1979 *Phys. Rev.* B **19** 320
Haldane F D 1992 *Helv. Phys. Acta* **65** 152 (*Preprint* cond-mat/0505529)
Castro Neto A H and Fradkin E 1994 *Phys. Rev.* B **49** 10877
Houghton A and Marston B 1993 *Phys. Rev.* B **48** 7790
Kwon H J, Houghton A and Marston B 1995 *Phys. Rev.* B **52** 8002
Anderson P W and Khveshchenko D 1995 *Phys. Rev.* B **52** 16415
Dhar A, Mandal G and Suryanarayana N V 2006 *J. High Energy Phys.* JHEP01(2006)118
Dhar A and Mandal G 2006 *Preprint* hep-th/0603154
[44] Polychronakos A P 2005 *Preprint* hep-th/0502150
[45] Jackiw R and Polychronakos A P 1999 *Commun. Math. Phys.* **207** 107
Jackiw R, Nair V P, Pi S-Y and Polychronakos A P 2004 *J. Phys. A: Math. Gen.* **37** R327
[46] Susskind L 2001 *Preprint* hep-th/0101029
Polychronakos A P 2001 *J. High Energy Phys.* JHEP04(2001)011
[47] See for example, Berenstein D 2004 *J. High Energy Phys.* JHEP07(2004)018
Lin H, Lunin O and Maldacena J 2004 *J. High Energy Phys.* JHEP10(2004)025
Bernevig B A, Brodie J, Susskind L and Toumbas N 2001 *J. High Energy Phys.* JHEP02(2001)003
Gubser S and Rangamani M 2001 *J. High Energy Phys.* JHEP05(2001)141
Castro C 2004 *Gen. Rel. Grav.* **36** 2605
[48] Hasebe K 2005 *Phys. Rev. Lett.* **94** 206802
Hasebe K 2005 *Preprint* hep-th/0503162
Hasebe K 2006 *Preprint* hep-th/0606007